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Some Nonlinear Networks Suggested by Learning Theory

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SOME NONLINEAR NETWORKS SUGGESTED BY LEARNING THEORY

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PREFACE

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ABSTRACT

We introduce several systems of nonlinear difference-differential equations and prove oscillation and global ratio limit theorems for some of them. These systems can be interpreted as a learning theory, or alternatively as a nonstationary prediction theory whose goal is to discuss the prediction of individual events, in a fixed order, and at prescribed times. They can also be interpreted as cross-correlated flows on networks, or as deformations of a probabilistic graph.

Each system possesses an underlying geometry characterized by a semistochastic matrix, and we study the effect of this geometry on the system's limiting behavior as $t \to \infty$. We also investigate the effects which the ratios of solutions of our systems have on the outputs of each system. We show that the average output of each system is not a good index of the mechanism which characterizes its interactions, especially when this average is computed over long time intervals. In particular, the average output is linear whereas the interactions are nonlinear. A system is discussed whose interactions are always locally reversible but whose global interactions are irreversible or not depending on the inputs received by the system. We also find systems whose entropy decreases monotonically in time and connect this phenomenon with the process of learning in these systems.

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CHAPTER I

INTERPRETATION AND BASIC PROPERTIES

1. INTRODUCTION

The present work introduces several systems of nonlinear difference-differential equations and studies their basic properties. These systems are of interest from a variety of viewpoints. Some of these are sketched in the following paragraphs.

A. Systems of Nonlinear Functional Equations

In studying a nonlinear system, it is always of interest to connect this system in some way with a related linear system. The simplest kind of linear system is given by the equation

$$\dot{X} = AX, \tag{1.1}$$

where $X = (x_1, x_2, \ldots, x_n)$ is a real n-vector and $A = \| a_{ij} \|$ is a real $n \times n$ matrix of constant coefficients. This kind of system has been thoroughly studied ([2], [8], [13]). A natural generalization of (1. 1) is a system of the form

$$\dot{X} = A(t)X + B(t)X(t-\tau) + C(t), \tag{1.2}$$

where $X = (x_1, x_2, \ldots, x_n)$ is again a real n-vector, $A(t) = \| a_{ij}(t) \|$ and $B(t) = \| b_{ij}(t) \|$ are known $n \times n$ matrices of real but variable coefficients, τ is a fixed nonnegative number, and $C(t) = (I_1(t), \ldots, I_n(t))$ is a real n-vector of known inhomogeneous (i. e., "forcing") terms. Systems such as (1. 2) have also been subjected to intensive study in a variety of cases ([3], [9], [12], [15]).

The systems which we shall consider can be formally written in the form (1. 2), with the following all-important difference. The matrices A(t) and B(t) are no longer known functions of time. Rather, they are matrices of nonlinear functionals of the unknown vector function $X(\xi)$ at past times $\xi \in [-\tau, t]$. We can write A(t) and B(t) in the form

$$A(t) = \mathcal{A}(X|_{[-\tau,t]}) \equiv \| \mathcal{A}_{ij}(X|_{[-\tau,t]}) \|$$

and

$$B(t) = \mathcal{B}(X|_{[-\tau,t]}) \equiv \| \mathcal{B}_{ij}(X|_{[-\tau,t]}) \|,$$

where $\mathcal{A}$ and $\mathcal{B}$ are matrices of nonlinear functionals of the vector function $X|_{[-\tau,t]}$. The systems which we shall study therefore fall under the general heading of "systems of nonlinear functional equations", or "systems of difference-differential equations with nonlinear feedback".

The present study amounts to a special choice of $\mathcal{A}$ and $\mathcal{B}$ which assures that the system of (1. 2) has interesting properties. This choice is very explicit, and all our results use its special properties. For example, we shall find in a special case that

$$\mathcal{A}_{ij}(X/_{[-\tau,t]}) = -\alpha \delta_{ij}$$

and

$$\mathcal{B}_{ij}(X/_{[-\tau,t]}) = B_{ji} \left(\sum_{k=1}^{n} B_{jk} \right)^{-1}$$

where

$$B_{ji} = (1-\delta_{ji})\left(\gamma_{ji} + \beta \int_0^t e^{uv} x_j(v-\tau) x_i(v) dv\right),$$

and the constants α, β, u, and γ_{ji} are positive.

B. A Prediction Theory

Our system of nonlinear difference-differential equations can be interpreted as a prediction theory. The goal of this theory is to discuss the prediction of individual events, in a fixed order, and at prescribed times. The theory is not homogeneous in time. A system which produces random predictions at time $t = 0$ can be gradually transformed into a system whose predictions become deterministic as $t \to \infty$. The converse is also true. The factor which primarily determines if a system becomes random or deterministic in its predictions as $t \to \infty$ is the system's input vector function $C(t)$, as in (1. 2). $C(t)$ is the "environment" or "experience" of the system, and we shall make precise the statement that these systems "adapt to their environment" or "learn from experience". These properties will be discussed in Chapter 2.

C. A Nonlinear System whose Average Output is Linear

Several of the systems which we will study have the property that the average output $x(t) = \frac{1}{n} \sum_{k=1}^{n} x_k(t)$ is related to the average input $I(t) = \frac{1}{n} \sum_{k=1}^{n} I_k(t)$ by the <u>linear</u> difference-differential equation

$$\dot{x}(t) = -\alpha x(t) + \beta x(t-\tau) + I(t), \tag{1. 3}$$

subjected to the following constraints: (1) α and β are positive; (2) $I(t)$ and $x(t)$ are nonnegative; and (3) τ is nonnegative. Of special interest is the case where $\alpha > \beta$ since then $\lim_{t \to \infty} x(t) = 0$ if $I(t) \equiv 0$, for all $\tau \geq 0$.

Although the averages $I(t)$ and $x(t)$ obey a linear equation, the individual inputs $I_i(t)$ and outputs $x_i(t)$, $i = 1, 2, \ldots, n$, obey a <u>nonlinear</u> system of equations. An infinite set of nonlinear systems whose averages

obey (1.3) will now be described. Positive coefficients α and β are needed to describe even the average output $x(t)$ of this system, as in (1.3). At least one additional positive coefficient u will be needed to scale the rate with which the various $x_i(t)$ interact with one another. The geometrical pathways over which these interactions occur are characterized by an $n \times n$ matrix $P = \| p_{ij} \|$ whose entries satisfy $p_{ij} \geqslant 0$ and $\Sigma_{k=1}^{n} p_{ik} = 1$. Such a matrix P is called a <u>stochastic matrix</u>. We can define a nonlinear system whose average output obeys the linear equation (1.3) for any positive constants α, β, and u; any nonnegative constant τ; any stochastic matrix P; and any nonnegative and continuous input vector function $\mathcal{C}(t) = (I_1(t), I_2(t), \ldots, I_n(t))$ whose average $\frac{1}{n}\Sigma_{k=1}^{n} I_k(t)$ equals $I(t)$. Such a system is

$$\dot{x}_i(t) = -\alpha x_i(t) + \beta \sum_{k=1}^{n} x_k(t-\tau) y_{ki}(t) + I_i(t), \quad i = 1, 2, \ldots, n \tag{1.4}$$

$$y_{jk}(t) = \frac{p_{jk} z_{jk}(t)}{\sum_{m=1}^{n} p_{jm} z_{jm}(t)}, \quad j, k = 1, 2, \ldots, n \tag{1.5}$$

$$\begin{aligned} \dot{z}_{jk}(t) &= -u z_{jk}(t) + \beta x_j(t-\tau) x_k(t), && \text{if } p_{jk} > 0 \\ z_{jk}(t) &= 0, && \text{if } p_{jk} = 0 \end{aligned} \tag{1.6) (*}$$

The system (*) exhibits useful properties only if its initial data are properly chosen. The initial data of (*) are always chosen to be continuous and nonnegative. Moreover, we require $z_{jk}(0)$ to be positive whenever p_{jk} is positive. The following theorem, which will be proved in the next section, assures us that (*) is always well-defined if the initial data are chosen in this way.

THEOREM 1.1: Let (*) be given with arbitrary continuous and non-negative initial data such that $z_{jk}(0) > 0$ whenever $p_{jk} > 0$. Then the solution of (*) exists and is unique, continuously differentiable, and non-negative in $(0, \infty)$. Moreover, if the initial data of any variable is positive, then this variable is positive in $[0, \infty)$.

From Theorem 1.1 it follows readily that the average of all $x_i(t)$ defined by (1.4) obeys (1.3), as the next corollary shows.

COROLLARY 1.1: Let $x(t)$ be the average $\frac{1}{n}\Sigma_{k=1}^{n} x_k(t)$ of the solutions of the n equations given in (1.4). Then $x(t)$ obeys the linear equation

$$\dot{x}(t) = -\alpha x(t) + \beta x(t-\tau) + I(t) \tag{1.3}$$

where $I(t) = \frac{1}{n} \Sigma_{k=1}^{n} I_k(t)$.

PROOF: By the definition of (*), $z_{jk}(0) > 0$ if $p_{jk} > 0$, and thus by Theorem 1.1, $z_{jk}(t) > 0$ for all $t \geqslant 0$ if $p_{jk} > 0$. Since $p_{jk} \geqslant 0$ and $\Sigma_{m=1}^{n} p_{jm} = 1$, at least one p_{jm} is positive, $m = 1, 2, \ldots, n$. Thus at least one $z_{jm}(0)$ is positive, and hence $\Sigma_{m=1}^{n} p_{jm} z_{jm}(t) > 0$, $t \geqslant 0$. This means that the denominator of $y_{jk}(t) = p_{jk} z_{jk}(t) (\Sigma_{m=1}^{n} p_{jm} z_{jm}(t))^{-1}$ in (1.5) is always well-defined. Summing over k therefore gives the identity $\Sigma_{k=1}^{n} y_{jk}(t) = 1$ for all $t \geqslant 0$ and all $j = 1, 2, \ldots, n$. This fact shows that summing over $i = 1, 2, \ldots, n$ in (1.4) and dividing by n gives (1.3).

It is natural to ask what new information is found by going to the

trouble of studying the complicated nonlinear system (*) instead of merely the simple linear averages in (1.3). The following simple corollary shows that the individual variables $x_i(t)$ and $z_{jk}(t)$ themselves are not always of exceptional interest.

COROLLARY 1.2: If $\lim_{t\to\infty} x(t) = 0$, then

$$\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} z_{jk}(t) = 0 \quad \text{for all } i, j, \text{ and } k .$$

PROOF: Since $x_i \geq 0$, $0 \leq x_i \leq nx$, where $\lim_{t\to\infty} nx(t) = 0$. Thus $\lim_{t\to\infty} x_i(t) = 0$. By (1.6),

$$z_{jk}(t) = e^{-ut}\left(z_{jk}(0) + \beta\int_0^t e^{uv} x_j(v-\tau)x_k(v)\,dv\right),$$

where $\lim_{t\to\infty} x_j(t-\tau)x_k(t) = 0$ and $u > 0$. Hence $\lim_{t\to\infty} z_{jk}(t) = 0$.

Corollary 1.2 uses the nonnegativity of the solutions to show that the average $x(t)$ and the individual solutions $x_i(t)$ and $z_{jk}(t)$ sometimes have an essentially identical behavior as $t \to \infty$. The new features contained in (*) shall turn out to be contained in the limiting behavior of various ratios of solutions of (*) as $t \to \infty$, rather than in the solutions themselves. This behavior will often prove to be independent of the initial data of (*), just so long as this data is positive. Thus we shall consider global ratio limit theorems for our nonlinear systems.

We shall actually prove a more general theorem than Theorem 1.1 in the next section. Theorem 1.1 holds in the more general case where $P = \| p_{ij} \|$ satisfies the conditions $p_{ij} \geq 0$ and $\Sigma_{k=1}^{n} p_{ik} = 0$ or 1. Such a matrix P is called semistochastic. The average output $x(t)$ does not always obey (1.3) for a general semistochastic matrix, but it has nonetheless helpful linearity properties.

D. Cross-Correlated Flows on Probabilistic Graphs

The nonlinear system (*) can be given an interpretation as a kind of flow on a probabilistic graph or network([11]). A finite directed graph G is a triple (V, E, φ) consisting of a finite set $V = \{ v \}$ of vertices, a finite set $E = \{ e \}$ of directed edges, and a mapping φ from E to $V \times V$. If $\varphi(e) = (v_1, v_2)$, then the edge e is said to have v_1 as its initial vertex and v_2 as its terminal vertex. Labelling the vertices as $v_1, v_2, \ldots, v_n$ we denote the edge e such that $\varphi(e) = (v_i, v_j)$ by e_{ij}. This situation can be visualized as in Figure 1 ([6]). We now geometrically interpret (*) in terms of such a finite directed graph G. The solutions $x_i(t)$ and $z_{jk}(t)$ of (*) have indices i, j, and k which vary from 1 to n. To each index i, we associate a vertex v_i and to each ordered pair of indices (i, j), such as appear in the solution z_{ij}, we associate an edge e_{ij}. To each vertex v_i, we assign the solution $x_i(t)$ of (*), which we therefore call the i^{th} vertex function of (*), and to each edge e_{ij} we assign the function $y_{ij}(t)$, which we therefore call the ij^{th} edge function. These assignments can be geometrically visualized as in Figure 2. $x_i(t)$ is thought of as a process going on at v_i,

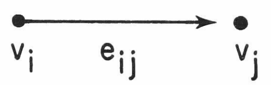


Fig. 1

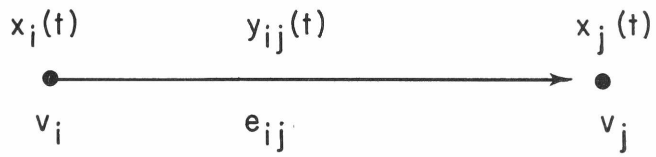


Fig. 2

and $y_{ij}(t)$ is thought of as a process going on at the arrowhead of e_{ij} . With this picture in mind, the equations of (*) can be interpreted as a kind of flow on the graph G in the following way.

1) <u>The Flow Along a Single Edge</u>: At every time $t - \tau$, a quantity of size $\beta x_i(t - \tau)$ leaves vertex v_i and flows along the edge e_{ij} at a finite velocity. This quantity reaches the arrowhead of e_{ij} at time t . When $\beta x_i(t - \tau)$ reaches the arrowhead of e_{ij} at time t , it activates the process described by $y_{ij}(t)$. As a result of this activation, a total magnitude $\beta x_i(t - \tau) y_{ij}(t)$ is emitted from the arrowhead and reaches vertex v_j at time t .

2) <u>The Total Flow Arriving at a Fixed Vertex</u>: The total flow received by vertex v_j from all other vertices v_i at time t is the <u>sum</u> of the flows received from each vertex v_i . By (1), this flow is

$$\beta \sum_{i=1}^{n} x_i(t-\tau) y_{ij}(t)$$

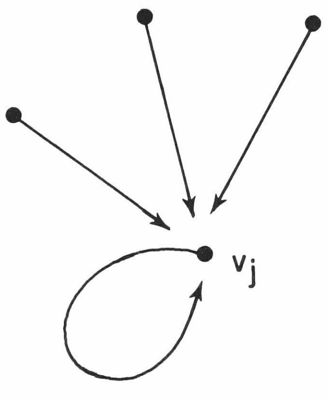


Fig. 3

(See Figure 3.) (1.4) says that the contribution of all vertices to the rate of change of the $x_j(t)$ process at v_j equals this total flow at every time t . The rate of change of $x_j(t)$ is also proportional to the magnitude of the input function $I_j(t)$, and $x_j(t)$ decays spontaneously at an exponential rate α .

3) <u>The Total Flow Leaving a Fixed Vertex</u>: By (1), the total flow received by all vertices v_i from a fixed vertex v_j at time t is

$$\beta \Sigma_{i=1}^{n} x_j(t-\tau) y_{ji}(t) = \beta x_j(t-\tau) \Sigma_{i=1}^{n} y_{ji}(t)$$

This flow is therefore either 0 or $\beta x_j(t-\tau)$, depending on whether $\Sigma_{i=1}^{n} p_{ji} = 0$ or 1. (See Figure 4.) Thus v_j either sends out no flow whatsoever at any time, or sends out a total flow which is proportional to its vertex function.

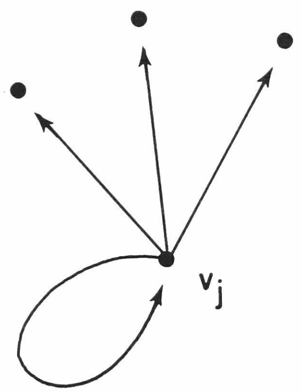


Fig. 4

4) <u>The Flow is Cross-Correlated</u>: The function $y_{ij}(t)$ which appears in the flow magnitude $\beta x_i(t-\tau)$ received by v_j from v_i at time t itself depends on the vertex functions, as is obvious from (1. 5) and (1. 6). The term $\beta x_i(t-\tau) x_j(t)$ appearing in (1. 6) has the following interpretation in terms of the flow along the edges. $\beta x_i(t-\tau)$ is the size of the flow received by the arrowhead of e_{ij} from v_i at time t. This arrowhead touches on v_j, whose vertex function has the value $x_j(t)$ at time t. $z_{ij}(t)$ <u>cross-correlates</u> the two quantities $\beta x_i(t-\tau)$ and $x_j(t)$ which impinge on the arrowhead at time t. That is, the rate of change of $z_{ij}(t)$ is proportional to $\beta x_i(t-\tau) x_j(t)$. $z_{ij}(t)$ also decays spontaneously at the rate u.

We form $y_{ij}(t)$ from the cross-correlating functions $z_{ij}(t)$ weighted by the coefficients p_{ij}; that is, from $p_{ij} z_{ij}(t)$. This is done by dividing $p_{ij} z_{ij}(t)$ by the sum of the functions $p_{ik} z_{ik}(t)$, $k = 1, 2, \ldots, n$, which belong to any edge e_{ik} that faces away from v_i, as in Figure 5. $y_{ij}(t)$ appears in t flow $\beta x_i(t-\tau) y_{ij}(t)$ instead of the unnormalized function $p_{ij} z_{ij}(t)$ to guarantee that the average output $x(t)$ of (*) obeys a <u>linear</u> equation.

By way of summary, the process (*) can be geometrically described

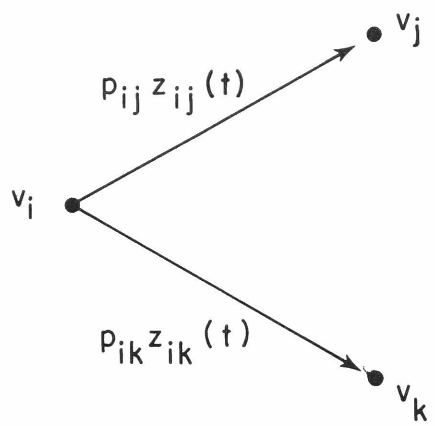


Fig. 5

as a directed flow on a graph or network. The magnitude of the flow at any time depends on the magnitude of the vertex functions at this time, on the normalized cross-correlations of the vertex functions at all past times, and on the inputs created by the experimenter.

5) Deforming a Probabilistic Graph: A closely related geometrical interpretation of (*) can be given in terms of the following definition. A finite directed probabilistic graph G_ψ is a pair (G, ψ) consisting of a finite directed graph G and a function $\psi : E \to [0, 1]$ such that $\psi(e_{ij}) \geqslant 0$ and $\Sigma_{k=1}^{n} \psi(e_{ik}) = 0$ or 1. ψ is called the weight function of G and $\psi_{ij} \equiv \psi(e_{ij})$ is called the weight of e_{ij}. At every time t, (*) can be thought of as a probabilistic graph $G(t)$ with weights $\psi_{ij}(t) \equiv y_{ij}(t)$. Then (*) becomes a 1-parameter family of probabilistic graphs $\mathcal{G} = \{G(t) : t \in [0, \infty)\}$. $\mathcal{G}$ can also be thought of as a continuously differentiable deformation of the probabilistic graph $G(0)$.

In terms of this geometrical picture of (*) as a deformation of a graph, we shall always have the following general question in mind. Given two probabilistic graphs G_0 and G_∞, does there exist an input vector function $\mathcal{J}(t) = (I_1(t), \ldots, I_n(t))$ which deforms $G(0) = G_o$ into $G(\infty) \equiv \lim_{t\to\infty} G(t) = G_\infty$? That is, starting with a process having transition probabilities G_o at time $t = 0$, do the fluctuations in these transition probabilities eventually converge to the stationary transition probabilities G_∞?

This question acquires special interest when P itself is realized as the weight function of a probabilistic graph; i.e., let $\psi_{ij} = p_{ij}$. Then $\mathcal{G}$ can be thought of as a "dynamical process" going on over the "geometrical framework" P. From this viewpoint, P is called the

coefficient matrix of $\mathcal{G}$, and the graph of Figure 6 is called the coefficient graph of $\mathcal{G}$. A natural question to ask now is the following one. When does the "geometry" P determine the "dynamics" $\mathcal{G}$ in this sense: when does $G(\infty) = P$? In Chapters 2, 3, and 5, we will treat cases in which P has little effect on $G(\infty)$. In Chapters 3 and 4 we will also treat a case in which P has a profound effect on $G(\infty)$.

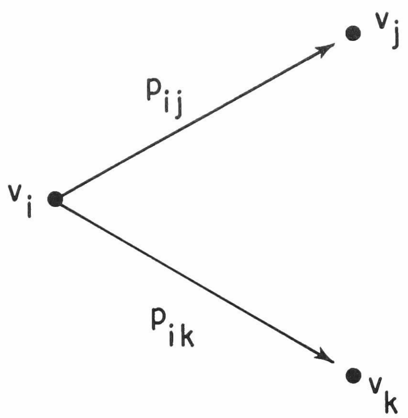


Fig. 6

E. Stability Properties are Graded in the Lag τ and the Number of Vertices n

In Chapter 4, we will study ratio limit theorems for the linearized system of (*) when

$$P = \begin{pmatrix} 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \vdots & & & & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 \end{pmatrix}$$

for any $n \geqslant 3$. We give sufficient conditions on the positive coefficients α, β, and u such that for fixed $\tau \geqslant 0$, the ratios always have limits if their initial data satisfy a mild technical condition. For fixed τ, it becomes easier to satisfy this condition as n increases. Thus we say stability properties of the linearized system are "graded" in n .

In Chapter 5, we will study ratio limit theorems for the linearized system of (*) when

$$P = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

for any $n \geqslant 2$. Here, for any fixed $n \geqslant 2$, it becomes easier to guarantee the existence of ratio limits as τ increases. Thus we say stability properties are "graded" in τ . The same is true for (*) itself.

2. POSITIVITY OF SOLUTIONS

In this section, we prove the following generalization of Theorem 1.1 as it was stated in Section 1C.

THEOREM 1.1: Let (*) be given with positive α, β, and u ; a nonnegative τ ; a semistochastic matrix P; and continuous and nonnegative functions $I_i(t)$. Let the initial data of (*) be continuous and nonnegative, and suppose $z_{jk}(0) > 0$ if $p_{jk} > 0$. Then the solution of (*) exists and is unique, continuously differentiable, and nonnegative in $(0, \infty)$. If the initial data of a given variable x_i or z_{jk} are positive, then this variable is positive in $[0, \infty)$.

PROOF. (*) can be written in vector form as

$$\dot{U}(t) = f(U(t), U(t-\tau)), \qquad (*)$$

with $U = (x_1, x_2, \ldots, x_n, z_{11}, z_{12}, \ldots, z_{n,n-1}, z_{nn})$,

$f = (f_1, f_2, \ldots, f_n, f_{11}, f_{12}, \ldots, f_{n,n-1}, f_{nn})$,

$f_i = -\alpha x_i + \beta \sum_{k=1}^n x_k(t-\tau) p_{ki} z_{ki} (\sum_{m=1}^n p_{km} z_{km})^{-1} + I_i$,

and $f_{jk} = (-u z_{jk} + \beta x_j(t-\tau) x_k)\theta(p_{jk})$,

where

$$\theta(p) = \begin{cases} 1 & \text{if} \quad p > 0 \\ 0 & \text{if} \quad p \leq 0 \,. \end{cases}$$

Let $\tau = 0$. Then $\dot{U}(t) = g(U(t))$, where $g(w) = f(w, w)$. By the continuity of g , a solution $U(t)$ exists in an interval with 0 as its left-hand endpoint.

If, moreover, $\| g(U^{(1)}) - g(U^{(2)}) \| \leq k(t) \| U^{(1)} - U^{(2)} \|$ for some continuous function $k(t)$ and any two solutions $U^{(1)}$ and $U^{(2)}$, then this interval is $(0, \infty)$ and the solution is unique and continuously differentiable ([7], p. 4). First we show that such a $k(t)$ exists if all x_i and z_{jk} are nonnegative. The only terms for which this is not obvious are the terms

$$x_j p_{jk} z_{jk} \left(\Sigma_{m=1}^{n} p_{jm} z_{jm} \right)^{-1}.$$

We use nonnegativity to estimate x_j above by a continuous function $m(t)$. By nonnegativity, $\dot{x}_i \geq -\alpha x_i$ and $\dot{z}_{jk} \geq -u z_{jk}$, or $x_i(t) \geq e^{-\alpha t} x_i(0)$ and $z_{jk}(t) \geq e^{-ut} z_{jk}(0)$. Thus $\Sigma_{m=1}^{n} p_{jm} z_{jm}(t) \geq e^{-ut} \Sigma_{m=1}^{n} p_{jm} z_{jm}(0) = \{0$ if $\Sigma_{m=1}^{n} p_{jm} = 0, > 0$ if $\Sigma_{m=1}^{n} p_{jm} = 1\}$. This implies $\Sigma_{m=1}^{n} y_{jm}(t) = \Sigma_{m=1}^{n} p_{jm} = 0$ or 1, from which we find that

$$\dot{x} \leq (\beta - \alpha) x + I,$$

where $x = \frac{1}{n} \Sigma_{k=1}^{n} x_k$ and $I = \frac{1}{n} \Sigma_{k=1}^{n} I_k$, or $x(t) \leq \frac{1}{n} m(t)$ where

$$m(t) = e^{(\beta-\alpha)t} \left(x(0) + \int_0^t e^{(\alpha-\beta)v} I(v)\, dv \right).$$

By nonnegativity, $x_j(t) \leq n x(t) \leq m(t)$. We can now prove the required Lipschitz condition. Obviously

$$\left| \frac{x_j^{(1)} p_{jk} z_{jk}^{(1)}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(1)}} - \frac{x_j^{(2)} p_{jk} z_{jk}^{(2)}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(2)}} \right| \leq |x_j^{(1)} - x_j^{(2)}| + m(t) \left| \frac{p_{jk} z_{jk}^{(1)}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(1)}} - \frac{p_{jk} z_{jk}^{(2)}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(2)}} \right|.$$

It therefore suffices to show that

$$p_{jk} \left| \frac{z_{jk}^{(1)}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(1)}} - \frac{z_{jk}^{(2)}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(2)}} \right| \leq h(t) \Sigma_{m=1}^{n} | z_{jm}^{(1)} - z_{jm}^{(2)} |$$

for some continuous $h(t)$. When $\Sigma_{m=1}^{n} p_{jm} = 0$, the choice $h(t) = 0$ suffices. Suppose $\Sigma_{m=1}^{n} p_{jm} = 1$. Then

$$p_{jk}\left|\frac{z_{jk}^{(1)}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(1)}} - \frac{z_{jk}^{(2)}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(2)}}\right|$$

$$= p_{jk}\left|\frac{(z_{jk}^{(1)} - z_{jk}^{(2)})\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(2)} + z_{jk}^{(2)}\Sigma_{m=1}^{n} p_{jm}(z_{jm}^{(2)} - z_{jm}^{(1)})}{(\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(1)})(\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(2)})}\right|$$

$$\leq p_{jk}\left\{\frac{|z_{jk}^{(1)} - z_{jk}^{(2)}| + \Sigma_{m=1}^{n} p_{jm}|z_{jm}^{(2)} - z_{jm}^{(1)}|}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(1)}}\right\}$$

$$\leq \frac{p_{jk}e^{ut}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(1)}(0)}\left(|z_{jk}^{(1)} - z_{jk}^{(2)}| + \Sigma_{m=1}^{n}|z_{jm}^{(1)} - z_{jm}^{(2)}|\right)$$

$$\leq \frac{2p_{jk}e^{ut}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(1)}(0)}\Sigma_{m=1}^{n}|z_{jm}^{(1)} - z_{jm}^{(2)}|.$$

Letting $h(t) = \dfrac{2p_{jk}e^{ut}}{\Sigma_{m=1}^{n} p_{jm} z_{jm}^{(1)}(0)}$ completes the proof when $\tau = 0$ except for the demonstration that x_i and z_{jk} are nonnegative.

By (1.6) and the nonnegativity of initial data, $z_{jk}(t)$ cannot become negative until either $x_j(t)$ or $x_k(t)$ becomes negative. Otherwise if $z_{jk}(t)$ is zero at $t = T_o$, then $\dot{z}_{jk}(T_o) = \beta x_j(T_o) x_k(T_o) \geqslant 0$. Let $t = T_1$ be the first zero of any function $x_i(t)$. Suppose in particular that $x_1(T_1) = 0$. Then by (1.4) $\dot{x}_1(T_1) = \beta\Sigma_{k=1}^{n} x_k(T_1) y_{k1}(T_1) + I_1(T_1) \geqslant 0$. x_1 can therefore never become negative, and all solutions are nonnegative. This completes the proof when $\tau = 0$.

Suppose $\tau > 0$. The existence of a solution of (*) follows by a standard "step-by-step" construction in each interval of the form $(n\tau, (n+1)\tau]$, $n = 0, 1, \ldots$ ([12]). To prove the remaining assertions, it suffices to show that $\| f(\xi_1, \eta) - f(\xi_2, \eta) \| \leq k(t) \| \xi_1 - \xi_2 \|$ for every η, and this can be done just as in the case $\tau = 0$.

CHAPTER II

GLOBAL RATIO LIMIT THEOREMS FOR OUTSTARS AND THEIR PREDICTION THEORETIC INTERPRETATION

PART I

1. Outstars

In this chapter, we discuss the simplest example of our prediction theory. This example is characterized by the coefficient matrix

$$P = \begin{pmatrix} 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ & & & & \\ & & \bigcirc & & \\ & & & & \end{pmatrix}.$$

The system therefore obeys the equations

$$\dot{x}_1(t) = -\alpha x_1(t) + I_1(t), \tag{2.1}$$

$$\dot{x}_j(t) = -\alpha x_j(t) + \beta x_1(t-\tau) y_{1j}(t) + I_j(t), \quad j = 2, 3, \ldots, n, \tag{2.2}$$

(*)

$$y_{1j}(t) = \frac{z_{1j}(t)}{\sum_{k=2}^{n} z_{1k}(t)}, \quad j = 2, 3, \ldots, n, \tag{2.3}$$

$$\dot{z}_{1j}(t) = -u z_{1j}(t) + \beta x_1(t-\tau) x_j(t), \quad j = 2, 3, \ldots, n \tag{2.4}$$

where all initial data are nonnegative and continuous, and moreover $z_{ij}(0) > 0$, $j \neq 1$, and $\Sigma_{k \neq 1} x_k(0) > 0$. The coefficient graph for (*) is given in Figure 7.

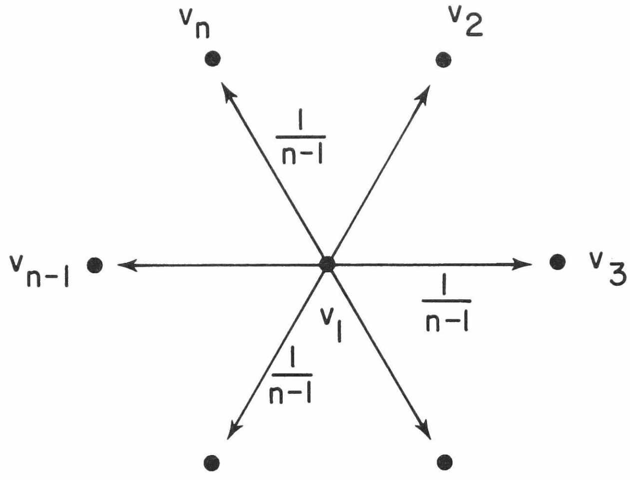


Fig. 7

(*) is therefore called an <u>outstar</u>. The vertex v_1 is called the <u>source</u> of the outstar and each vertex v_j, $j \neq 1$, is called a <u>sink</u> of the outstar. The set $B = \{v_j : j \neq 1\}$ of all sinks is called the <u>border</u> of the outstar.

In Part I of this chapter, we shall study (*) from a purely mathematical point of view. In Part II, we give these results a prediction theoretic interpretation. Our mathematical discussion will concern itself with the limiting behavior of (*) as $t \to \infty$ for special choices of the input vector function $C(t) = (I_1(t), I_2(t), \ldots, I_n(t))$. These choices will be interpreted in Part II as the presentation to the machine which (*) represents of sequences of predictions to be learned.

The choices of $C(t)$ will be divided into three general cases. In the first case, no inputs reach the border of the outstar at any time. In the second case, inputs do reach this border and continue to do so for all time. In the third case, inputs do reach the border but only for a finite amount of time. All of these cases can be treated by a single method. The

success of this method depends on the fact that (*) can be transformed into a more tractable system of equations expressed in terms of new unknown variables. These variables can be classified into two classes. The first class consists of sums over all vertices $j \neq 1$ in the border of (*). These sums are $x^{(1)} = \Sigma_{k \neq 1} x_k$, $z^{(1)} = \Sigma_{k \neq 1} z_{1k}$, and $I^{(1)} = \Sigma_{k \neq 1} I_k$. The second class consists of three 1-parameter families of probability distributions associated with (*). These are $X = \{X_j : j \neq 1\}$, $y = \{y_{1j} : j \neq 1\}$, and $\theta = \{\theta_j : j \neq 1\}$, where

$$X_j = \frac{x_j}{x^{(1)}}, \quad y_{1j} = \frac{z_{1j}}{z^{(1)}}, \quad \text{and} \quad \theta_j = \frac{I_j}{I^{(1)}}.$$

We shall find that the sums $x^{(1)}$ and $z^{(1)}$ over the border depend on time only through the known inputs I_1 and $I^{(1)}$. In particular, they are independent of the unknown probabilities X and y. Moreover, (*) can be replaced by a system of equations for the time evolution of the probability distributions X, y, and θ. The coefficients in these equations depend only on I_1 and the known sums $x^{(1)}$, $z^{(1)}$, and $I^{(1)}$. These facts are summarized in the following two lemmas.

LEMMA 2.1. The source function x_1 and the sums $x^{(1)}$ and $z^{(1)}$ depend on time only through the known inputs I_1 and $I^{(1)}$.

PROOF. The assertion is obvious for x_1 by (2.1). Sum (2.2) over $j \neq 1$ using the fact that $\Sigma_{j \neq 1} y_{1j} = 1$. Then

$$\dot{x}^{(1)}(t) = -\alpha x^{(1)}(t) + \beta x_1(t-\tau) + I^{(1)}(t), \tag{2.5}$$

and so by (2.1) the assertion is obvious for $x^{(1)}$. Summing (2.4) over $j \neq 1$ we find

$$\dot{z}^{(1)}(t) = -uz^{(1)}(t) + \beta x_1(t-\tau)\,x^{(1)}(t), \tag{2.6}$$

which gives the assertion for $z^{(1)}$ by (2. 1) and (2. 5)

LEMMA 2. 2. (*) can be transformed into the following system of equations for the probability distributions y and X .

$$\dot{X}_j = A_1(y_{1j} - X_j) + B_1(\theta_j - X_j)$$

and

$$\dot{y}_{1j} = C_1(X_j - y_{1j}),$$

where $A_1(t) = \dfrac{\beta x_1(t-\tau)}{x^{(1)}(t)}$, $B_1(t) = \dfrac{I^{(1)}(t)}{x^{(1)}(t)}$, and $C_1(t) = \beta x_1(t-\tau)x^{(1)}(t)/z^{(1)}(t)$.

PROOF. (2. 7) has the following derivation. Since $X_j = \dfrac{x_j}{x^{(1)}}$,

$$\dot{X}_j = \frac{1}{x^{(1)}}\left(\dot{x}_j - x_j\,\frac{\dot{x}^{(1)}}{x^{(1)}}\right).$$

Substituting (2. 2) and (2. 5) into this equation gives

$$\dot{X}_j = \frac{1}{x^{(1)}}\Bigg[-\alpha x_j + \beta x_1(t-\tau)\,y_{1j} + I_j - x_j\left(-\alpha + \frac{\beta x_1(t-\tau) + I^{(1)}}{x^{(1)}}\right)\Bigg]$$

$$= \frac{\beta x_1(t-\tau)}{x^{(1)}}\left(y_{1j}-X_j\right)+\frac{I^{(1)}}{x^{(1)}}\left(\theta_j-X_j\right)$$

$$= A_1\left(y_{1j}-X_j\right)+B_1\left(\theta_j-X_j\right).$$

(2.8) is derived in the following way. Since $y_{1j} = z_{1j}/z^{(1)}$,

$$\dot{y}_{1j} = \frac{1}{z^{(1)}}\left(\dot{z}_{1j} - z_{1j}\frac{\dot{z}^{(1)}}{z^{(1)}}\right).$$

Substituting (2.4) and (2.6) into this equation gives

$$\dot{y}_{1j} = \frac{1}{z^{(1)}}\left[-uz_{1j}+\beta x_1(t-\tau)x_j - z_{1j}\left(-u+\frac{\beta x_1(t-\tau)x^{(1)}}{z^{(1)}}\right)\right]$$

$$= \frac{\beta x_1(t-\tau)x^{(1)}}{z^{(1)}}\left(y_{1j}-X_j\right)$$

$$= C_1\left(y_{1j}-X_j\right).$$

2. Outstars with an Input-free Border

We use Lemmas 2.1 and 2.2 to study the case in which no inputs reach the border of the outstar at any time. Thus $I_j(t) \equiv 0$, $j \neq 1$, and we say the border of (*) is input-free. The main fact needed to carry out our prediction theory in this case is the following.

THEOREM 2.1. If $x_1 \not\equiv 0$, then y_{1j} and X_j are monotone in opposite senses and $\lim_{t \to \infty} y_{1j}(t) = \lim_{t \to \infty} X_j(t)$. If $x_1 \equiv 0$, then y_{1j} and X_j are constant.

PROOF. By (2.7) and the hypothesis $I^{(1)}(t) \equiv 0$,

$$\dot{X}_j = A_1 (y_{1j} - X_j), \tag{2.7}$$

where $A_1(t) = \dfrac{\beta x_1(t-T)}{x^{(1)}(t)}$ is nonnegative. By (2.8)

$$\dot{y}_{1j} = C_1 (X_j - y_{1j}), \tag{2.8}$$

where $C_1(t) = \dfrac{\beta x_1(t-T)\, x^{(1)}(t)}{z^{(1)}(t)}$ is nonnegative. From (2.7) and (2.8) we draw the following conclusions. If $x_1 \equiv 0$ then y_{1j} and X_j are constant since $\dot{y}_{1j} = \dot{X}_j \equiv 0$. Suppose that $x_1 \not\equiv 0$. If $X_j(t_o) = y_{1j}(t_o)$, then $X_j(t) = y_{1j}(t) =$ constant for all $t \geqslant t_o$. By (2.1), there is a $T_o < \infty$ such that $x_1(t - T) = 0$ for $t \in [0, T_o]$ and $x_1(t - T) > 0$ for $t > T_o$. If $X_j(0) > y_{1j}(0)$, then $X_j(t)$ and $y_{1j}(t)$ are constant for $t \in [0, T_o]$. $X_j(t)$ is strictly monotone decreasing and $y_{1j}(t)$ is strictly monotone increasing

for all $t \in (T_o, T_1)$, where T_1 is the smallest root, if any, of the equation $X_j(t) = y_{1j}(t)$. If such a T_1 exists, then $X_j(t)$ and $y_{1j}(t)$ are constant for $t > T_1$. We shall show in the next paragraph that no such T_1 exists. If no such T_1 exists, then $X_j(t)$ decreases monotonically for all $t > T_o$ and $y_{1j}(t)$ increases monotonically for all $t > T_o$. Since X_j and y_{1j} are bounded, the limits $Q_j = \lim_{t\to\infty} X_j(t)$ and $P_{1j} = \lim_{t\to\infty} y_{1j}(t)$ exist. If $X_j(0) < y_{1j}(0)$, the same argument goes through with all inequalities reversed. In all cases, therefore, X_j and y_{1j} are monotone in opposite senses and $|X_j - y_{1j}|$ is monotone nonincreasing.

We now show that T_1 does not exist and that $P_{1j} = Q_j$ if $x_1 \not\equiv 0$. Subtracting (2.7) from (2.8) gives

$$(y_{1j} - X_j)^{\cdot} = - D_1 (y_{1j} - X_j), \tag{2.9}$$

where $D_1 = A_1 + C_1 = \beta x_1(t - T)\left(\frac{1}{x^{(1)}} + \frac{x^{(1)}}{z^{(1)}}\right)$. Integrating (2.9) gives

$$y_{1j}(t) - X_j(t) = (y_{1j}(0) - X_j(0))\,\Omega_1(t), \tag{2.9'}$$

where $\Omega_1(t) = \exp\left[-\beta \int_0^t x_1(\xi - T)\left(\frac{1}{x^{(1)}(\xi)} + \frac{x^{(1)}(\xi)}{z^{(1)}(\xi)}\right) d\xi\right]$. To show that T_1 does not exist, note that $\Omega_1(t) > 0$, $t \geqslant 0$. Thus $y_{1j}(0) \neq X_j(0)$ implies $y_{1j}(t) \neq X_j(t)$. To show that $P_{1j} = Q_j$, we must show that $\lim_{t\to\infty} \Omega_1(t) = 0$, or that

$$\lim_{t\to\infty} \int_0^t x_1(\xi - \tau)\left(\frac{1}{x^{(1)}(\xi)} + \frac{x^{(1)}(\xi)}{z^{(1)}(\xi)}\right) d\xi = \infty .$$

Since $x^{(1)}/z^{(1)}$ is positive, it suffices to show that

$$\lim_{t\to\infty} \int_0^t \frac{x_1(\xi-\tau)}{x^{(1)}(\xi)}\, d\xi = \infty .$$

For $t > 0$,

$$\int_0^t \frac{x_1(\xi-\tau)}{x^{(1)}(\xi)}\, d\xi = \int_0^t \frac{x_1(\xi-\tau)\,d\xi}{e^{-\alpha\xi}\left(x^{(1)}(0)+\beta\int_0^\xi x_1(v-\tau)e^{\alpha v}dv\right)}$$

$$= \int_0^t \frac{d}{d\xi}\log\left(x^{(1)}(0)+\beta\int_0^\xi x_1(v-\tau)e^{\alpha v}dv\right)$$

$$= \log\left(1+\frac{\beta}{x^{(1)}(0)}\int_0^t x_1(\xi-\tau)e^{\alpha\xi}d\xi\right).$$

By (2.1), $\dot{x}_1 \geqslant -\alpha x_1$. Thus $\displaystyle\int_0^t \frac{x_1(\xi-\tau)}{x^{(1)}(\xi)}\, d\xi$ diverges at a logarithmic rate as $t \to \infty$, and $P_{1j} = Q_j$, $j \neq 1$.

Theorem 2.1 contains all the information we shall need about an outstar with input-free border to discuss our prediction theory. It can be tersely summarized by Figure 8.

Theorem 2.1 shows that the limits $\lim_{t\to\infty} X_j(t)$ and $\lim_{t\to\infty} y_{1j}(t)$ do not vary continuously as a function of the initial data $x_1(v)$, $v \in [-\tau, 0]$. This theorem is picturesquely called the "speck of dust" theorem because it describes an alternative which depends on whether or not the source function x_1

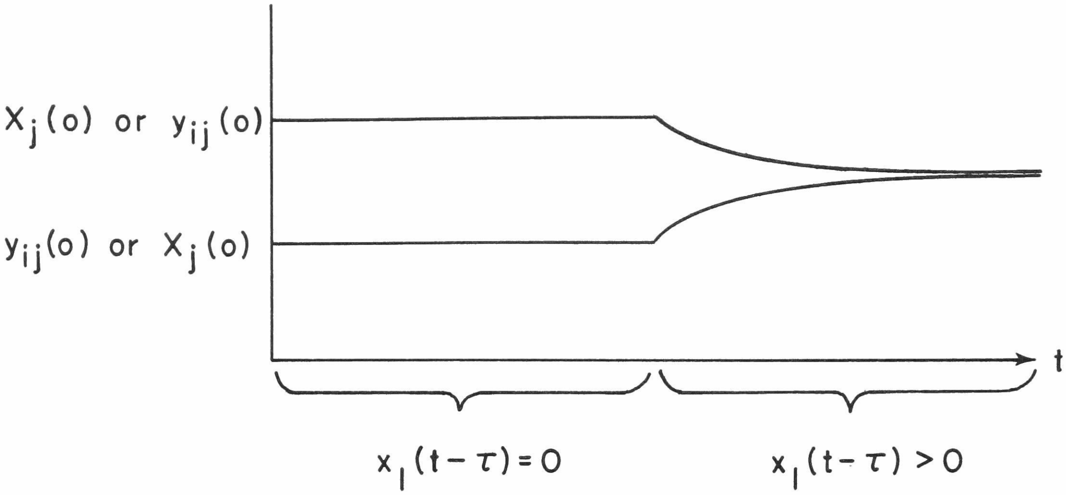


Fig. 8

is identically zero. Since I_1 is nonnegative, the positivity of $x_1(t_o)$ for any t_o implies the positivity of $x_1(t)$ for all $t \geqslant t_o$. Thus if the initial data of x_1 are identically zero, then $x_1(t)$ remains zero until a positive value of $I_1(t)$, no matter how small — that is, a "speck of dust" — reaches the source v_1. Thereafter $x_1(t)$ remains positive at all times.

By Theorem 2.1, if $X_j(0) = y_{1j}(0)$ then $X_j(t) = y_{1j}(t) =$ constant for all $t \geqslant 0$. This means in particular that arbitrary probability distributions can arise as limits $\lim_{t\to\infty} X_j(t) = \lim_{t\to\infty} y_{1j}(t)$, $j \neq 1$. The coefficient matrix P of an outstar thus does not uniquely determine the limiting distributions when the border of the outstar is input-free. This answers the question of Section D-5 in Chapter I in the negative for an outstar.

More information is available concerning an outstar with input-free border than is contained in Theorem 2.1, because (*) can be explicitly integrated in this case to give precise information about the relative rates at which the probability distributions associated with different vertices and edges approach their limits as $t \to \infty$. It can be shown that $x_j(t)$ and $z_{1j}(t)$ obey equations of the form

$$x_j(t) = X_j(0)a(t) + (y_{1j}(0) - X_j(0))b(t)$$

and

$$z_{1j}(t) = X_j(0)c(t) + (y_{1j}(0) - X_j(0))d(t),$$

where a , b , c , and d are nonnegative and continuous functions that depend on I_1 and the initial data $x_1(0)$, $x^{(1)}(0)$, and $z^{(1)}(0)$. Moreover, $a = x^{(1)}$ and $c = z^{(1)}$, so that

$$X_j(t) - X_j(0) = (y_{1j}(0) - X_j(0))\frac{b(t)}{a(t)} \qquad (2.10)$$

and

$$y_{1j}(t) - X_j(0) = (y_{1j}(0) - X_j(0))\frac{d(t)}{c(t)} \qquad (2.11)$$

for all $j = 1, 2, \ldots, n$. Since also $x_1(-\tau) \neq 0$ implies b and d are positive, (2.10) and (2.11) show that if $x_1(-\tau) \neq 0$, $y_{1i}(0) \neq X_i(0)$, and $y_{1j}(0) \neq X_j(0)$, then

$$\frac{y_{1i}(t) - X_i(0)}{y_{1j}(t) - X_j(0)} = \frac{X_i(t) - X_i(0)}{X_j(t) - X_j(0)} = \frac{y_{1i}(0) - X_i(0)}{y_{1j}(0) - X_j(0)} \quad \text{for all } t \geq 0 .$$

Thus the probability distributions at different vertices approach their limits at the same rate, except for a multiplicative factor that depends on their initial data.

3. Outstars Whose Border Never Becomes Input-Free

In the preceding section, we found that any probability distribution $y_{1j}(t) = X_j(t)$, $j \neq 1$, remains constant for all $t \geq 0$ when the outstar's border is input-free. This fact provides an affirmative answer to the following question. Given any probability distribution θ_j , $j \neq 1$, does there exist an input vector function $C = (I_1, \ldots, I_n)$ for which $X_j(0) = y_{1j}(0) = \theta_j$ and $\lim_{t\to\infty} X_j(t) = \lim_{t\to\infty} y_{1j}(t) = \theta_j$? Any nonnegative I with $I_j \equiv 0$, $j \neq 1$, accomplishes this goal. A natural generalization of this question is the following question. Given any three probability distributions $\theta_j^{(1)}$, $\theta_j^{(2)}$, and $\theta_j^{(3)}$, $j \neq 1$, does there exist an input vector function C for which $X_j(0) = \theta_j^{(1)}$, $y_{1j}(0) = \theta_j^{(2)}$, and $\lim_{t\to\infty} X_j(t) = \lim_{t\to\infty} y_{1j}(t) = \theta_j^{(3)}$? We now answer this question in the affirmative and provide a considerable amount of supplementary information concerning the manner in which the probability distributions X_j and y_{1j} approach their limits. We do this in the following theorem.

<u>THEOREM 2. 2.</u> Suppose the inputs to the border of an outstar have the form $I_j(t) = \theta_j I(t)$, $j \neq 1$, where $\{ \theta_j : j \neq 1 \}$ is a fixed, but arbitrary, probability distribution, and $I_1(t)$ and $I(t)$ are arbitrary nonnegative and continuous functions. Then the functions $f_j(t) = y_{1j}(t) - X_j(t)$, $g_j(t) = X_j(t) - \theta_j$, and $\dot{y}_{1j}(t)$ change sign at most once, and not at all if $f_j(0)g_j(0) \geq 0$. Moreover, $f_j(0)g_j(0) > 0$ implies $f_j(t)g_j(t) > 0$ for all $t \geq 0$. Suppose furthermore that $I_1(t)$ and $I(t)$ are bounded functions such that $I(t) \not\to 0$ as $t \to \infty$ and also that there exist two positive constants C and T_o for which

$$\int_0^t e^{-\alpha(t-\xi)} I_1(\xi)\,d\xi \geq C \quad \text{for } t \geq T_0 .$$

Then $\lim\limits_{t\to\infty} X_j(t) = \lim\limits_{t\to\infty} y_{1j}(t) = \theta_j \ , \ j \neq 1$.

PROOF. The proof is divided into three steps. In step (I) we prove that f_j , g_j , and $\mathring{y}_{1j}$ change sign at most once and thus, that $\lim\limits_{t\to\infty} y_{1j}(t)$ exists, $j \neq 1$. In step (II) , the existence of these limits along with estimates of $C_1(t)$ and $\mathring{C}_1(t)$ for large t are used to show that the limits $\lim\limits_{t\to\infty} X_j(t)$ exist and equal the limits $\lim\limits_{t\to\infty} y_{1j}(t)$. In step (III), the common value of these limits is shown to be θ_j by estimating $A_1(t)$, $B_1(t)$, $\dot{A}_1(t)$, and $\dot{B}_1(t)$ for large t .

(I). Subtracting (2. 7) from (2. 8) gives

$$\mathring{f}_j = - D_1 f_j + B_1 g_j \ . \tag{2. 12}$$

Since $(X_j - \theta_j)^{\bullet} = \dot{X}_j$, (2. 7) may be written as

$$\dot{g}_j = - B_1 g_j + A_1 f_j \ . \tag{2. 13}$$

Equations (2. 12) and (2. 13) are special cases of the following simple but basic lemma.

LEMMA 2. 3. Let the functions f and g satisfy the differential equations

$$\dot{f} = af + bg$$

and

$$\dot{g} = cf + dg$$

where a , b , c , and d are continuous functions and the off-diagonal coefficients b and c are nonnegative. Then f and g change sign at most once and not at all if $(fg)(0) \geq 0$. Moreover, $(fg)(0) > 0$ implies $(fg)(t) > 0$ for all $t \geq 0$.

Lemma 2.3 can be geometrically visualized by Figure 9,

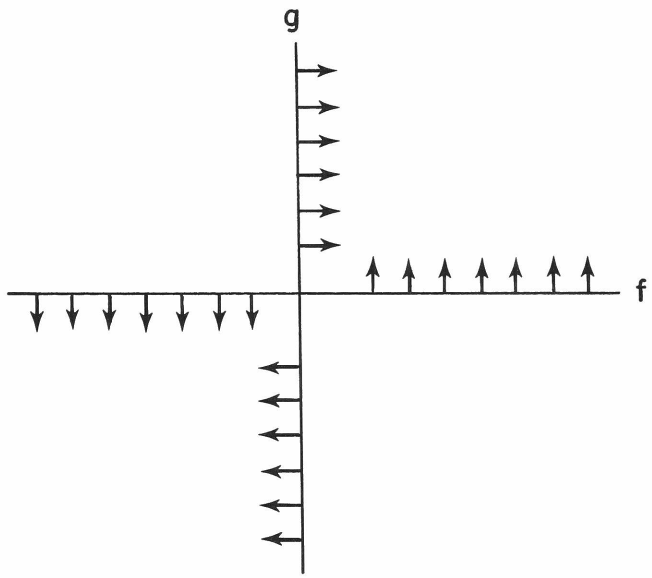


Fig. 9

which shows the (f, g) plane. The direction of the arrows indicates the path of the (f, g) point through time.

PROOF. Clearly $(fg)^{\bullet} = (a + d)fg + bg^2 + cf^2 \geq (a + d)fg$ by the nonnegativity of b and c . Thus for any $t_o \geq 0$, $(fg)(t_o) \geq 0$ implies

$(fg)(t) \geqslant 0$ for all $t \geqslant t_o$. More can be said. Let $(fg)(t_o) > 0$, where $f(t_o) > 0$ (say), and let $t = t_1 > t_o$ be the first zero of f , or g , or both (of f , say). Then f and g are both nonnegative in $[t_o, t_1]$, so $\dot{f} \geqslant af$ in $[t_o, t_1]$, and $0 = f(t_1) > e^{a(t_1 - t_o)} f(t_o) > 0$, which is a contradiction. Thus $(fg)(t_o) > 0$ implies $(f_g)(t) > 0$ for all $t \geqslant t_o$.

Only the case $(fg)(0) < 0$ remains, where $f(0) < 0$ (say). Then either $f(t) < 0 < g(t)$ for all $t \geqslant 0$, or there is a first $t_1 > 0$ when f , or g , or both have a zero. If such a t_1 exists, we are in a previous case, so that f and g change sign at most once.

Lemma 2.3 can be directly applied to (2.12) and (2.13) by letting $f = f_j$, $g = g_j$, $a = -D_1$, $b = B_1$, $c = A_1$, and $d = -B_1$. We conclude that f_j and g_j change sign at most once and not at all if $(f_j g_j)(0) \geqslant 0$. Moreover, $(f_j g_j)(0) > 0$ implies $(f_j g_j)(t) > 0$ for all $t \geqslant 0$.

By (2.8),

$$\dot{y}_{1j} = -C_1 f_j \ .$$

Since C_1 is nonnegative, $\dot{y}_{1j}$ also changes sign at most once and not at all if $(f_j g_j)(0) \geqslant 0$. In particular, there exists a T_1 such that $y_{1j}(t)$ is a monotonic function for $t \geqslant T_1$. y_{1j} is also bounded and continuous. Thus $\lim_{t \to \infty} y_{1j}(t)$ exists for all $j \neq 1$.

(II). Using the facts proved in (I), we now show that the limits $\lim_{t \to \infty} X_j(t)$ exist. The first step in this proof is to establish various estimates for the coefficients

$$A_1(t) = \frac{\beta x_1(t-\tau)}{x^{(1)}(t)} ,$$

$$B_1(t) = \frac{I^{(1)}(t)}{x^{(1)}(t)} = \frac{I(t)}{x^{(1)}(t)},$$

and

$$C_1(t) = \frac{\beta x_1(t-\tau)x^{(1)}(t)}{z^{(1)}(t)}$$

which appear in (2. 7) and (2. 8). The purpose of these estimates is to show that $\ddot{y}_{1j}(t)$ is bounded for sufficiently large t. This fact, in turn, will be needed to prove that $\lim_{t\to\infty} \dot{y}_{1j}(t) = 0$, from which it will follow with the help of the estimates that $\lim_{t\to\infty} X_j(t)$ exists and equals $\lim_{t\to\infty} y_{1j}(t)$.

The estimates needed for A_1, B_1, and C_1 are the following. We shall find positive constants λ_i, $i = 1, 2, 3, 4, 5$ and a time T_2, such that for $t \geqslant T_2$, the inequalities $\lambda_1 \leqslant C_1(t) \leqslant \lambda_2$, $A_1(t) \leqslant \lambda_3$, $B_1(t) \leqslant \lambda_4$, and $|\dot{C}_1(t)| \leqslant \lambda_5$ hold. To establish these estimates, we make comparable estimates on the functions x_1, $x^{(1)}$, and $z^{(1)}$ from which A_1, B_1, and C_1 are constructed. Firstly we establish lower bounds for these functions for large t.

By hypothesis, there exist positive constants C and T_o such that

$$\int_0^t e^{-\alpha(t-\xi)} I_1(\xi)\, d\xi \geqslant C \quad \text{for } t \geqslant T_o.$$

Thus by integrating (2. 1), we find

$$x_1(t) \geqslant e^{-\alpha t} x(0) + C$$
$$\geqslant C, \quad t \geqslant T_0 .$$

Substituting this inequality into the integrated form of (2.5) gives for $t \geqslant 2(T_0 + \tau)$,

$$x^{(1)}(t) \geqslant e^{-\alpha t}\left(x^{(1)}(0) + \int_0^{T_0+\tau} e^{\alpha\xi}\left(\beta x_1(\xi-\tau) + I^{(1)}(\xi)\right) d\xi\right.$$
$$\left. + \beta C \int_{T_0+\tau}^{t} e^{\alpha\xi} d\xi\right)$$
$$\geqslant \frac{\beta C}{\alpha}\left(1 - e^{-\alpha(T_0+\tau)}\right)$$
$$\equiv d > 0 .$$

Substituting these inequalities into the integrated form of (2.6) gives for $t \geqslant 3(T_0 + \tau)$,

$$z^{(1)}(t) \geqslant e^{-ut}\left(z^{(1)}(0) + \beta\int_0^{2(T_0+\tau)} e^{u\xi} x_1(\xi-\tau) x^{(1)}(\xi)\, d\xi\right.$$
$$\left. + \beta C d \int_{2(T_0+\tau)}^{t} e^{u\xi} d\xi\right)$$
$$\geqslant \frac{\beta C d}{u}\left(1 - e^{-u(T_0+\tau)}\right)$$
$$\equiv e > 0 .$$

Upper bounds for x_1, $x^{(1)}$, and $z^{(1)}$ follow by the boundedness of I_1 and $I^{(1)}$. Letting $M_1 = \sup\{I_1(t) : t \geq 0\}$ and $M^{(1)} = \sup\{I(t) : t \geq 0\}$, we readily find that

$$x_1(t) \leq x_1(0) + \frac{1}{\alpha} M_1 \equiv M < \infty,$$

$$x^{(1)}(t) \leq x^{(1)}(0) + \frac{\beta}{\alpha}(M + M^{(1)}) \equiv N < \infty,$$

and

$$z^{(1)}(t) \leq z^{(1)}(0) + \frac{\beta M N}{u} \equiv R < \infty .$$

Let $T_2 = 3(T_0 + \tau)$. Then the following definitions of the λ_i, $i = 1, 2, 3, 4,$ obviously suffice for $t \geq T_2$. $\lambda_1 = \frac{\beta C d}{R}$,

$\lambda_2 = \frac{\beta M N}{e}$, $\lambda_3 = \frac{\beta M}{d}$, and $\lambda_4 = \frac{M^{(1)}}{d}$.

$\dot{x}_1$, $\dot{x}^{(1)}$, and $\dot{z}^{(1)}$ can also be shown to be bounded by simple estimates of the above kind. Using these estimates along with those derived above readily shows that there exists a $\lambda_5 < \infty$ such that $|\dot{C}_1(t)| < \lambda_5$ for all $t \geq T_2$.

These various estimates on the functions A_1, B_1, C_1, and $\dot{C}_1$ suffice to show that $\ddot{y}_{1j}(t)$ is bounded for $t \geq T_2$, since by (2.7) and (2.8),

$$
\begin{aligned}
|\ddot{y}_{1j}| &= |\dot{c}_1(X_j - y_{1j}) + c_1(\dot{X}_j - \dot{y}_{1j})| \\
&\leq 2|\dot{c}_1| + |c_1| \left[(A_1 + C_1)|y_{1j} - X_j| + B_1|\theta_j - X_j| \right] \\
&\leq 2\lambda_5 + \lambda_2 \left(2(A_1 + C_1) + 2B_1 \right) \\
&\leq 2\left(\lambda_5 + \lambda_2 (\lambda_2 + \lambda_3 + \lambda_4) \right) \\
&< \infty
\end{aligned}
$$

for $t \geqslant T_2$.

To show that $\lim_{t \to \infty} \dot{y}_{1j}(t) = 0$, we need the following lemma.

LEMMA 2. 3. Suppose $f(t) \to \alpha < \infty$ as $t \to \infty$ and $\ddot{f}$ is bounded. Then $\dot{f}(t) \to 0$ as $t \to \infty$.

PROOF. Suppose not. Then for some $\epsilon > 0$, there exists a sequence $\{t_n\}$ with $\lim_{n \to \infty} t_n = \infty$ such that $|\dot{f}(t_n)| \geqslant \epsilon$ for all n . We can suppose $\dot{f}(t_n) \geqslant \epsilon$ for all n without loss of generality. Since $\ddot{f}$ is bounded, there exists a δ such that $\dot{f} \geqslant \frac{\epsilon}{2}$ on infinitely many nonoverlapping intervals $I_n = [U_n, U_n + \delta]$ of length δ , where $\lim_{n \to \infty} U_n = \infty$.

Thus $f(U_n + \delta) - f(U_n) \geqslant \frac{\epsilon\delta}{2}$ for all n , and $f \not\to \alpha < \infty$ as $t \to \infty$,

which is a contradiction.

Replacing f by y_{1j} in Lemma 2. 3 immediately shows that $\lim_{t \to \infty} \dot{y}_{1j}(t) = 0$.

We can now show the existence of $\lim_{t\to\infty} X_j(t)$. By (I) we can assume that $\dot{y}_{1j}(t) \geq 0$ and thus that $(X_j - y_{1j})(t) \geq 0$ for $t \geq T_o$ without loss of generality. Also $C_1(t) \geq \lambda_1 > 0$ for $t \geq T_2$ Thus by (2.8)

$$\begin{aligned} \dot{y}_{1j} &= C_1(X_j - y_{1j}) \\ &\geq \lambda_1(X_j - y_{1j}) \\ &\geq 0 \end{aligned}$$

for $t \geq \max(T_o, T_2)$. Since $\lim_{t\to\infty} \dot{y}_{1j}(t) = 0$, it follows immediately that $\lim_{t\to\infty}(X_j(t) - y_{1j}(t)) = 0$. We also know that $\lim_{t\to\infty} y_{1j}(t)$ exists, by (I). Thus $Q_j \equiv \lim_{t\to\infty} X_j(t)$ exists and equals $\lim_{t\to\infty} y_{1j}(t)$.

(III). The existence and equality of the limits $\lim_{t\to\infty} y_{1j}$ and $\lim_{t\to\infty} X_j$ can now be used to show that this common limit Q_j equals θ_j . Proceeding as in (II), it is easy to show that $\ddot{X}_j$ is bounded. Since $\lim_{t\to\infty} X_j(t) < \infty$ exists, Lemma 2.3 implies that $\lim_{t\to\infty} \dot{X}_j(t) = 0$. Also by (II), $\lim_{t\to\infty}(y_{1j}(t) - X_j(t)) = 0$ and A_1 is bounded. Thus by (2.7), $\lim_{t\to\infty} B_1(t)(\theta_j - X_j(t)) = 0$. Since $Q_j = \lim_{t\to\infty} X_j(t)$, we find $(\theta_j - Q_j)\cdot \lim_{t\to\infty} B_1(t) = 0$, and either $Q_j = \theta_j$ or $\lim_{t\to\infty} B_1(t) = 0$. Since $B_1 = I/x^{(1)}$, the inequality $x^{(1)} \leq N$ from (I) gives $B_1 \geq \frac{1}{N} I \geq 0$. By hypothesis, $I(t) \not\to 0$ as $t \to \infty$. Thus $B_1(t) \not\to 0$ as $t\to\infty$, and we conclude that $Q_j = \theta_j$, which completes the proof.

The following corollary to Theorem 2.2 will have a useful interpretation. An <u>input pulse</u> $J(t)$ is a nonnegative and continuous function which is positive in a finite interval (λ_1, λ_2) , where $0 \leq \lambda_1 < \lambda_2$. The corollary describes what happens when $I_1(t)$ and $I(t)$ are chosen to be periodic successions of input pulses.

COROLLARY 2.1. Let the functions $I_1(t)$ and $I(t)$ of Theorem 2.2 be defined as follows.

$$I_1(t) = \sum_{k=0}^{\infty} J_1(t - k(w+W))$$

and

$$I(t) = \sum_{k=0}^{\infty} J_2(t - w - k(w+W)),$$

where $J_i(t)$ is an arbitrary input pulse which is positive in the interval $(0, \lambda_i)$, $\lambda_i > 0$, $i = 1, 2$, and w and W are nonnegative numbers whose sum is positive. Then all the conclusions of Theorem 2.2 hold.

PROOF. It is obvious that I_1 and I are nonnegative, continuous, and bounded functions. It is also obvious that I does not converge to zero as $t \to \infty$. It remains only to find positive constants C and T_0 such that

$$\Phi(t) \equiv \int_0^t e^{-\alpha(t-\xi)} I_1(\xi)\, d\xi \geq C \quad \text{for } t \geq T_0.$$

Writing $p = w + W$, let

$$F(t) = \int_{t-p}^{t} e^{-\alpha(t-\xi)} I_1(\xi)\, d\xi, \quad t \geq p.$$

Then for any $n \geq 1$ and $t \in [np, (n+1)p)$,

$$\Phi(t) \geq F(t) + e^{-\alpha p} F(t-p) + \cdots + e^{-\alpha(n-1)p} F(t-(n-1)p).$$

Clearly $F(t) \geq F(t-p)$ for all $t \geq 2p$, since $I_1(t) \geq I_1(t-p)$. Thus

$$\Phi(t) \geq \left(1+e^{-\alpha p}+\cdots+e^{-\alpha(n-1)p}\right) F(t-(n-1)p)$$

$$\geq F(t-(n-1)p),$$

for any $t \in [np, (n+1)p)$. Since $f_n(t) \equiv F(t-(n-1)p)$ is a positive and continuous function of $t \in [p, 2p]$, letting $C = \inf\{f_n(t) : t \in [p, 2p]\}$ (> 0) and $T_0 = p$ completes the proof.

4. Outstars whose Border Eventually Becomes Input-free

In the previous section, we considered outstars subjected to inputs of the form

$$I_1(t) = \sum_{k=0}^{\infty} J_1(t-k(w+W))$$

and

$$I_j(t) = \theta_j \sum_{k=0}^{\infty} J_2(t-w-k(w+W)), \quad j \neq 1.$$

Since each vertex of such an outstar receives infinitely many input pulses, we denote the outstar by $G^{(\infty)}$ and affix the superscript "(∞)" to each of its functions. For example, we write I_1 as $I_1^{(\infty)}$, x_j as $x_j^{(\infty)}$, and so on.

The border of $G^{(\infty)}$ never becomes input-free. In the present section, we consider outstars whose border does eventually become input free. Given any outstar of type $G^{(\infty)}$, we shall construct an infinite

sequence of outstars $G^{(1)}$, $G^{(2)}$, ..., $G^{(N)}$, ..., each with the same initial data as $G^{(\infty)}$, and each with a border which eventually becomes input-free. We shall then study the limiting behavior of the functions of $G^{(N)}$ as N and t are permitted to become large by comparing these functions with those of $G^{(\infty)}$ and of outstars with input-free borders. To simplify our discussion, we shall assume that $J_1 = J_2 = J$, where $J(t)$ is positive in $(0, \lambda)$, $\lambda > 0$. The same method can be applied when the input pulses J_1 and J_2 are not the same.

Given an outstar of type $G^{(\infty)}$ and a positive integer N, the outstar $G^{(N)}$ is defined by the following prescriptions: (1) $G^{(N)}$ has the same initial data as $G^{(\infty)}$, and (2) the input functions of $G^{(N)}$ are

$$I_1^{(N)}(t) = \sum_{k=0}^{N-1} J(t-k(w+W))$$

and

$$I_j^{(N)}(t) = \theta_j \sum_{k=0}^{N-1} J(t-w-k(w+W)), \quad j \neq 1.$$

That is, the first N input pulses received by the vertices of $G^{(N)}$ are the same as the first N input pulses received by the vertices of $G^{(\infty)}$. Thereafter no input pulses occur in $G^{(N)}$, so that the border of $G^{(N)}$ is eventually input-free. $G^{(N)}$ is called the N-<u>truncation of</u> $G^{(\infty)}$ because its functions agree with those of $G^{(\infty)}$ in the interval $[0, N(w+W)]$. Denote the X_j and y_{1j} functions of $G^{(N)}$ by $X_j^{(N)}$ and $y_{1j}^{(N)}$, respectively. The following theorem holds for these functions.

4A) The Probability Distributions of an Outstar $G^{(N)}$ Remain Essentially Fixed for Large Times

THEOREM 2.3. Let $G^{(1)}, G^{(2)}, \ldots, G^{(N)}, \ldots$ be the sequence of N-truncations of any outstar of type $G^{(\infty)}$ with arbitrary nonnegative and continuous initial data such that $z_{1j}(0) > 0$, $j \neq 1$. Then

(1) For every $N = 1, 2, \ldots,$ the limits $\lim_{t \to \infty} X_j^{(N)}(t)$ and $\lim_{t \to \infty} y_{1j}^{(N)}(t)$ exist and are equal.

(2) For every $N = 1, 2, \ldots,$ and $j \neq 1$, the functions $f_j^{(N)} = y_{1j}^{(N)} - X_j^{(N)}$, $g_j^{(N)} = X_j^{(N)} - \theta_j$, and $\dot{y}_{1j}^{(N)}$ change sign at most once and not at all if $f_j^{(N)}(0)g_j^{(N)}(0) \geq 0$. Moreover, $f_j^{(N)}(0)g_j^{(N)}(0) > 0$ implies $f_j^{(N)}(t)g_j^{(N)}(t) > 0$ for all $t \geq 0$.

(3) For every $N = 1, 2, \ldots,$ and all $t \geq w + \lambda + (N - 1)(w + W)$, $X_j^{(N)}(t)$ and $y_{1j}^{(N)}(t)$ are contained in an interval $[m_j^{(N)}, M_j^{(N)}] \subset (0, 1)$, where

$$\lim_{N \to \infty} m_j^{(N)} = \lim_{N \to \infty} M_j^{(N)} = \theta_j \quad \text{for all } j \neq 1 .$$

In particular,

$$\lim_{N \to \infty} \lim_{t \to \infty} X_j^{(N)}(t) = \lim_{N \to \infty} \lim_{t \to \infty} y_{1j}^{(N)}(t) = \theta_j , \quad j \neq 1 .$$

Before proving the theorem, we illustrate its claim pictorially for the special case $\theta_j = \delta_{j2}$ in two outstars $G^{(M)}$ and $G^{(N)}$ with $M << N$ in Figure 10.

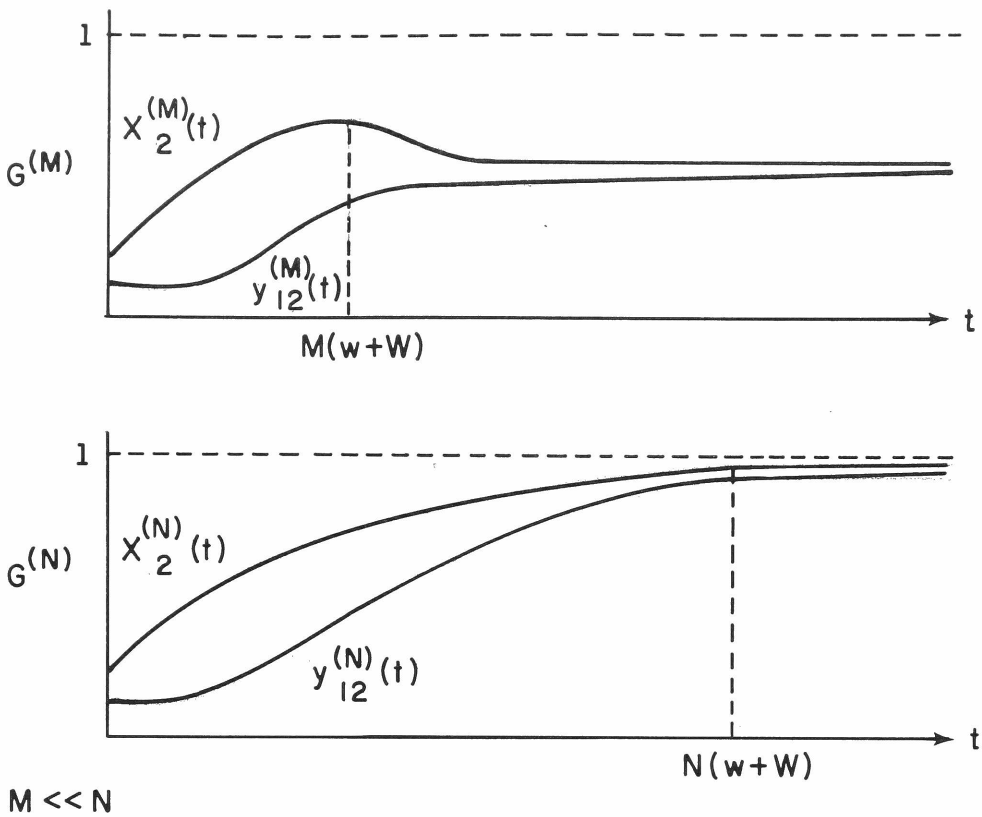


Fig. 10

PROOF. We shall carry out the proof in the case $\theta_j = \delta_{j2}$. The same method goes through in general, but it is more tedious. The idea of the proof is to try to divide the time interval $[0, \infty)$ of each $G^{(N)}$ into two parts $[0, \gamma_N)$ and $[\gamma_N, \infty)$, where $\lim_{N \to \infty} \gamma_N = \infty$. In $[0, \gamma_N)$, the functions of $G^{(N)}$ agree with those of $G^{(\infty)}$ and we can apply Theorem 2. 2 to them. In $[\gamma_N, \infty)$, $G^{(N)}$ has an input-free border and we can apply Theorem 2. 1 to its functions within this interval. This goal can be accomplished with but one technical reservation which appears in Case 2 below.

Clearly $X_j^{(N)}(t) = X_j^{(\infty)}(t)$ and $y_{1j}^{(N)}(t) = y_{1j}^{(\infty)}(t)$ for $t \in [0, N(w + W)]$.

In particular, $X_j^{(N)}(N(w+W)-v) = X_j^{(\infty)}(N(w+W)-v)$ and $y_{1j}^{(N)}(N(w+W)-v) = y_{1j}^{(\infty)}(N(w+W)-v)$ for any fixed $v \in [0, N(w+W)]$.

By Corollary 2.1, $\lim_{t\to\infty} X_j^{(\infty)}(t) = \lim_{t\to\infty} y_{1j}^{(\infty)}(t) = \delta_{j2}$. Hence

$$\lim_{N\to\infty} X_j^{(N)}(N(w+W)-v) = \lim_{N\to\infty} y_{1j}^{(N)}(N(w+W)-v) = \delta_{j2} \tag{2.14}$$

for any fixed $v \geqslant 0$.

In every $G^{(N)}$, no input pulses occur during the time interval $[\lambda(N), \infty)$, where $\lambda(N) = w + (N-1)(w+W) + \lambda$. In particular, $G^{(N)}$ has an input-free border in $[\lambda(N), \infty)$, and we can therefore apply the results of Section 3 in this time interval.

By Theorem 2.1, $y_{1j}^{(N)}(t)$ and $X_j^{(N)}(t)$ are monotonic in opposite senses and $\left| y_{1j}^{(N)}(t) - X_j^{(N)}(t) \right|$ decreases monotonically to zero for $t \geqslant \lambda(N)$. Letting

$$M_j^{(N)} = \max\left\{ y_{1j}^{(N)}(\lambda(N)),\ X_j^{(N)}(\lambda(N)) \right\} \quad \text{and} \quad m_j^{(N)} = \min\left\{ y_{1j}^{(N)}(\lambda(N)),\ X_j^{(N)}(\lambda(N)) \right\},$$

we conclude in particular that $y_{1j}^{(N)}(t)$ and $X_j^{(N)}(t)$ are contained in the interval $[m_j^{(N)}, M_j^{(N)}]$ for all $t \geqslant \lambda(N)$. We must now distinguish two cases.

<u>Case 1.</u> $\lambda(N) \leqslant N(w+W)$:

Letting $v = N(w+W) - \lambda(N) \geqslant 0$ in (2.14), we conclude that

$$\lim_{N\to\infty} m_j^{(N)} = \lim_{N\to\infty} M_j^{(N)} = \delta_{j2} .$$

Since $y_{1j}^{(N)}(t)$ and $X_j^{(N)}(t)$ are contained in $[m_j^{(N)}, M_j^{(N)}]$ for all $t \geqslant \lambda(N)$, we can, by taking N sufficiently large, bring $y_{1j}^{(N)}(t)$ and $X_j^{(N)}(t)$ as close to δ_{j2} as we please. Moreover, these functions will remain there for all large t, even though no input pulses occur for $t \geqslant \lambda(N)$. In particular, by Theorem 2.1, we conclude

$$\lim_{t\to\infty} y_{1j}^{(N)}(t) = \lim_{t\to\infty} X_j^{(N)}(t) \in [m_j^{(N)}, M_j^{(N)}],$$

so that

$$\lim_{N\to\infty}\lim_{t\to\infty} y_{1j}^{(N)}(t) = \lim_{N\to\infty}\lim_{t\to\infty} X_j^{(N)}(t) = \delta_{j2}.$$

Case 2. $\lambda(N) > N(w + W)$:

In this case, the conclusion of Case 1 still holds, but we need more information to reach it. By (2.7) and (2.8),

$$\dot{X}_2^{(N)} = A_1^{(N)}\left(y_{12}^{(N)} - X_2^{(N)}\right) + B_1^{(N)}\left(1 - X_2^{(N)}\right) \tag{2.7}$$

and

$$\dot{y}_{12}^{(N)} = C_1^{(N)}\left(X_2^{(N)} - y_{12}^{(N)}\right), \tag{2.8}$$

where $A_1^{(N)}$, $B_1^{(N)}$, and $C_1^{(N)}$ are nonnegative. If $\dot{y}_{12}^{(N)} \leq 0$, then by (2.8), $y_{12}^{(N)} - x_2^{(N)} \geq 0$ and so, by (2.7), $\dot{x}_2^{(N)} \geq 0$. If $\dot{y}_{12}^{(N)} \geq 0$, then by (2.8) $x_2^{(N)} \geq y_{12}^{(N)}$. In either case, it is clear that $y_{12}^{(N)}(t)$ and $x_2^{(N)}(t)$ exceed $\min(x_2^{(N)}(t_o), y_{12}^{(N)}(t_o))$ for any t and t_o in $[N(w + W), \lambda(N)]$ with $t \geq t_o$. Letting $t = \lambda(N)$ and $t_o = N(w + W)$, we find that $y_{12}^{(N)}(\lambda(N))$ and $x_2^{(N)}(\lambda(N))$ exceed $\min(x_2^{(N)}(N(w + W)), y_{12}^{(N)}(N(w + W)))$. By (2.14),

$$\lim_{N\to\infty} x_2^{(N)}(N(w+W)) = \lim_{N\to\infty} y_{12}^{(N)}(N(w+W)) = 1.$$

Since $x_2^{(N)}$ and $y_{12}^{(N)}$ are bounded above by 1, this implies

$$\lim_{N\to\infty} x_2^{(N)}(\lambda(N)) = \lim_{N\to\infty} y_{12}^{(N)}(\lambda(N)) = 1.$$

Since $G^{(N)}$ has an input-free border in $[\lambda(N), \infty)$, it now follows just as in Case 1 that $x_2^{(N)}(t)$ and $y_{12}^{(N)}(t)$ can be brought as close to 1 as we desire by taking N sufficiently large, and will thereafter remain there.

The conclusions of (2) follow simply by pasting together the results from Theorems 2.1 and 2.2. That is, we consider $G^{(N)}$ to be a $G^{(\infty)}$ for small times and an outstar with input-free border for large times.

We can summary Theorem 2.3 in the following way. As N is taken increasingly large, the probability distributions $x_j^{(N)}$ and $y_{1j}^{(N)}$ of $G^{(N)}$ approximate the δ_{j2} distribution (or more generally any fixed probability distribution θ_j) with increasingly good accuracy for all large t.

4B) The Outputs of Each $G^{(N)}$ Decay Exponentially for Large Times.

We shall now show that in each $G^{(N)}$, the behavior of the output functions $x_j^{(N)}(t)$ differs radically from the behavior of the ratios $X_j^{(N)}(t)$ as $t \to \infty$. In Theorem 2.3 we showed that the ratios $X_j^{(N)}(t)$ remain essentially fixed for large times t in each $G^{(N)}$. Now we show that the output functions $x_j^{(N)}(t)$ decay to zero at an exponential rate as $t \to \infty$. This contrast between ratios $X_j^{(N)}$ and outputs $x_j^{(N)}$ will have an important prediction theoretic meaning.

PROPOSITION 2.1. In each $G^{(N)}$, the outputs $x_j^{(N)}$ decay exponentially to zero as $t \to \infty$.

PROOF. Since $G^{(N)}$ is input free in $[\lambda(N), \infty)$, (2.1) implies that

$$\dot{x}_1^{(N)}(t) = -\alpha x_1^{(N)}(t) \quad \text{for} \quad t \geq \lambda(N) .$$

Thus $x_1^{(N)}$ converges exponentially to zero as $t \to \infty$. Similarly for $j \neq 1$ and $t \geq \lambda(N) + \tau$,

$$\begin{aligned} \dot{x}_j^{(N)}(t) &= -\alpha x_j^{(N)}(t) + \beta x_1^{(N)}(t-\tau) y_{1j}^{(N)}(t) \\ &\leq -\alpha x_j^{(N)}(t) + \beta x_1^{(N)}(t-\tau), \end{aligned}$$

where $x_1^{(N)}(t - \tau)$ converges exponentially to zero. Thus $x_j^{(N)}$ converges exponentially to zero as well.

4C The Effect of Fixed Ratios on Outputs.

In (4B) we showed that even though the ratios $X_j^{(N)}(t)$ remain essentially fixed for large t, the outputs $x_j^{(N)}(t)$ decay exponentially to zero as $t \to \infty$. It therefore seems that the absolute magnitudes of the ratios and the outputs are completely unrelated as $t \to \infty$. This is not always true, but we must change our outstars $G^{(N)}$ slightly to see this. We do this only in the case $\theta_j = \delta_{j2}$ for simplicity.

The input functions of $G^{(N)}$ are in this case

$$I_1^{(N)}(t) = \sum_{k=0}^{N-1} J(t - k(w+W))$$

and

$$I_j^{(N)}(t) = \delta_{j2} \sum_{k=0}^{N-1} J(t - w - k(w+W)).$$

In particular, $G^{(N)}$ has an input-free border in $[\lambda(N), \infty)$. Let $f_N(t)$ be any nonnegative and continuous function which is positive only in the interval $[\lambda(N), \infty)$. Such an f_N is called admissible. Given any sequence $f = (f_1, f_2, \ldots, f_N, \ldots)$ of admissible f_N's, we shall now construct a sequence $G^{(1,f)}, G^{(2,f)}, \ldots, G^{(N,f)}, \ldots$ of outstars that is closely related to the sequence $G^{(1)}, G^{(2)}, \ldots, G^{(N)}, \ldots$ of outstars. For each $N = 1, 2, \ldots, G^{(N,f)}$ is defined in terms of $G^{(N)}$ by the following prescriptions: (1) The initial data of $G^{(N,f)}$ are the same as that of $G^{(N)}$ (and hence that of $G^{(\infty)}$); (2) The input functions of $G^{(N,f)}$ are

$$I_1^{(N,f)} = I_1^{(N)} + f_N$$

and

$$I_j^{(N,f)} = I_j^{(N)} \quad , \quad j \neq 1 \ .$$

We say that the sequence $G^{(1,f)}, G^{(2,f)}, \ldots$ is derived from f and $G^{(\infty)}$. Any derived sequence of this kind obeys the following theorem.

THEOREM 2.3f. Let $G^{(1,f)}, G^{(2,f)}, \ldots, G^{(N,f)}, \ldots$ be the sequence of outstars derived from any admissible f and any $G^{(\infty)}$ with arbitrary nonnegative and continuous initial data such that $z_{1j}(0) > 0$, $j \neq 1$. Then all the conclusions of Theorem 2.3 hold for this sequence with superscripts "(N, f)" replacing superscripts "(N)".

PROOF. Since $I^{(N,f)} = I_1^{(N)} + f_N$, the functions of $G^{(N,f)}$ agree with those of $G^{(N)}$ in $[0, N(w+W)]$. Since $I_j^{(N,f)} = I_j^{(N)}$, $j \neq 1$, $G^{(N,f)}$ has an input-free border in $[\lambda(N), \infty)$. The rest of the proof is now just as in Theorem 2.3.

We now consider special choices of f which show some of the effects which the fixed ratios $X_j^{(N,f)}(t)$ can have on the outputs $x_j^{(N,f)}(t)$ for large t.

(1) $f_N(t) = J(t - \Lambda(N))$, where $\Lambda(N) >> \lambda(N)$. For this choice of f_N, $G^{(N,f)}$ differs from $G^{(N)}$ only in the occurrence of an input pulse $J(t - \Lambda(N))$ at the source of $G^{(N,f)}$ at time $t = \Lambda(N)$. In particular, $G^{(N,f)}$ is the same as $G^{(N)}$ in $[0, \Lambda(N)]$, and so $G^{(N,f)}$ is input-free in $[\lambda(N), \Lambda(N)]$. By Proposition 2.1, the outputs $x^{(N,f)}(t)$ decay exponentially towards zero for $t \in [\lambda(N), \Lambda(N)]$. Since $\Lambda(N) >> \lambda(N)$, we can assume that all the outputs $x^{(N,f)}(t)$ are very small at time $t = \Lambda(N)$, and we write $x_j^{(N,f)}(\Lambda(N)) \cong 0$, $j = 1, 2, \ldots, n$. This

is true for every $N \geqslant 1$.

By Theorem 2.3f, we can, by taking N sufficiently large, guarantee that $y_{1j}^{(N,f)}(t)$ approximates δ_{j2} as closely as we wish for $t \geqslant \lambda(N)$. In particular, we can write $y_{1j}^{(N,f)}(t) \cong \delta_{j2}$ for $t \geqslant \Lambda(N)$. We are now ready to discuss the effects which the input pulse $f_N(t) = J(t - \Lambda(N))$ has on the outputs of $G^{(N,f)}$.

The first fact of interest is that $f_N(t)$ has essentially no effect whatsoever on the outputs $x^{(N,f)}$, $j \neq 1, 2$. By (2.2), we have for $t \geqslant \Lambda(N)$ and $j \neq 1, 2$ that

$$\dot{x}_j^{(N,f)}(t) = -\alpha x_j^{(N,f)} + \beta x_1^{(N,f)}(t-\tau)\, y_{1j}^{(N,f)}(t)$$

$$\cong -\alpha x_j^{(N,f)} + \beta x_1^{(N,f)}(t-\tau) \cdot 0$$

$$= -\alpha x_j^{(N,f)} .$$

Thus

$$x_j^{(N,f)}(t) \cong e^{-\alpha(t-\Lambda(N))}\, x_j^{(N,f)}(\Lambda(N))$$

$$\cong 0 .$$

By contrast, $f_N(t)$ has a substantial effect on the output $x_2^{(N,f)}$.
By (2. 2), we have for $t \geq \Lambda(N)$

$$\dot{x}_2^{(N,f)}(t) = -\alpha x_2^{(N,f)}(t) + \beta x_1^{(N,f)}(t-\tau)\, y_{12}^{(N,f)}(t)$$

$$\cong -\alpha x_2^{(N,f)}(t) + \beta x_1^{(N,f)}(t-\tau)$$

and thus

$$x_2^{(N,f)}(t) \cong x_2^{(N,f)}(\Lambda(N))\, e^{-\alpha(t-\Lambda(N))} + \beta \int_{\Lambda(N)}^{t} e^{-\alpha(t-v)} x_1^{(N,f)}(v-\tau)\, dv$$

$$\cong \beta \int_{\Lambda(N)}^{t} e^{-\alpha(t-v)} x_1^{(N,f)}(v-\tau)\, dv \qquad (2.15)$$

But by (2. 1) we have for $t \geq \Lambda(N)$ that

$$\dot{x}_1^{(N,f)}(t) = -\alpha x_1^{(N,f)}(t) + J(t-\Lambda(N)),$$

and thus

$$x_1^{(N,f)}(t) = x_1^{(N,f)}(\Lambda(N))\, e^{-\alpha(t-\Lambda(N))} + \int_{\Lambda(N)}^{t} e^{-\alpha(t-v)} J(v-\Lambda(N))\, dv$$

$$\cong \int_{\Lambda(N)}^{t} e^{-\alpha(t-v)} J(v-\Lambda(N))\,dv$$

$$= e^{-\alpha(t-\Lambda(N))} \int_{0}^{t-\Lambda(N)} e^{\alpha v} J(v)\,dv.$$

(2.16)

Substituting (2.16) into (2.15) gives for $t \in [\Lambda(N), \Lambda(N) + \tau]$ that

$$x_1^{(N,f)}(t) \cong 0$$

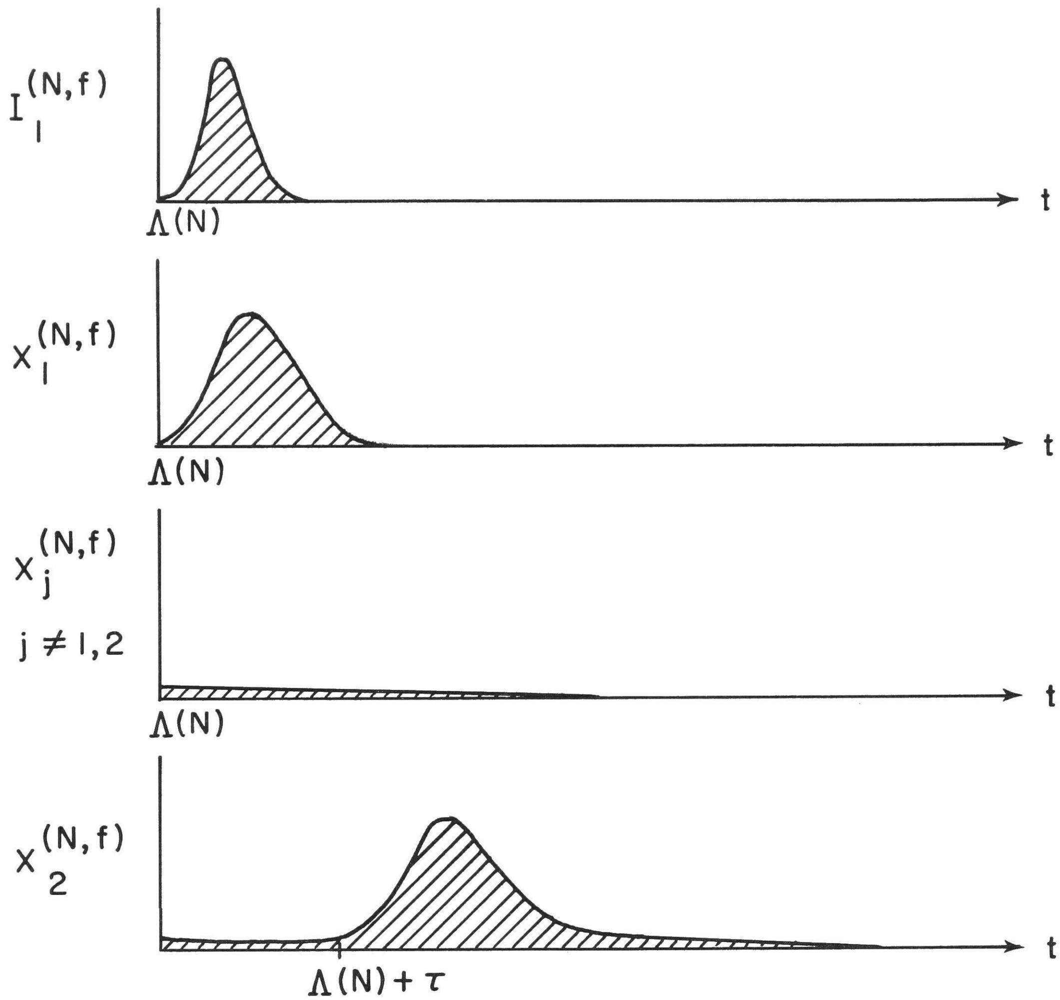
and for $t \geq \Lambda(N) + \tau$ that

$$x_2^{(N,f)}(t) \cong \beta \int_{\Lambda(N)+\tau}^{t} e^{-\alpha(t-v)} x_1^{(N,f)}(v-\tau)\,dv$$

$$= \beta e^{-\alpha(t-\Lambda(N)-\tau)} \int_{0}^{t-\Lambda(N)-\tau} dv \int_{0}^{v} e^{\alpha w} J(w)\,dw.$$

Thus $x_2^{(N,f)}(t)$ grows substantially in the interval $(\Lambda(N) + \tau, \Lambda(N) + \tau + \lambda)$ and then decays once again at an exponential rate to zero.

We summarize these statements in Figure 11. These facts can be stated heuristically as follows. If the vertices v_1 and v_2 are each perturbed periodically by N input pulses, where N is a large number, then a later test input pulse to v_1 creates a large output only from v_2. The periodic input pulses to the vertices v_1 and v_2 channel most of the mass $\Sigma_{k=2}^{n} y_{1k}^{(N,f)}(t)$ of the edges e_{1k} into the edge e_{12}, and then e_{12}



$I_1^{(N,f)}$
t
$\Lambda(N)$
$X_1^{(N,f)}$
t
$\Lambda(N)$
$X_j^{(N,f)}$
$j \neq 1,2$
t
$\Lambda(N)$
$X_2^{(N,f)}$
t
$\Lambda(N)+\tau$

Fig. 11

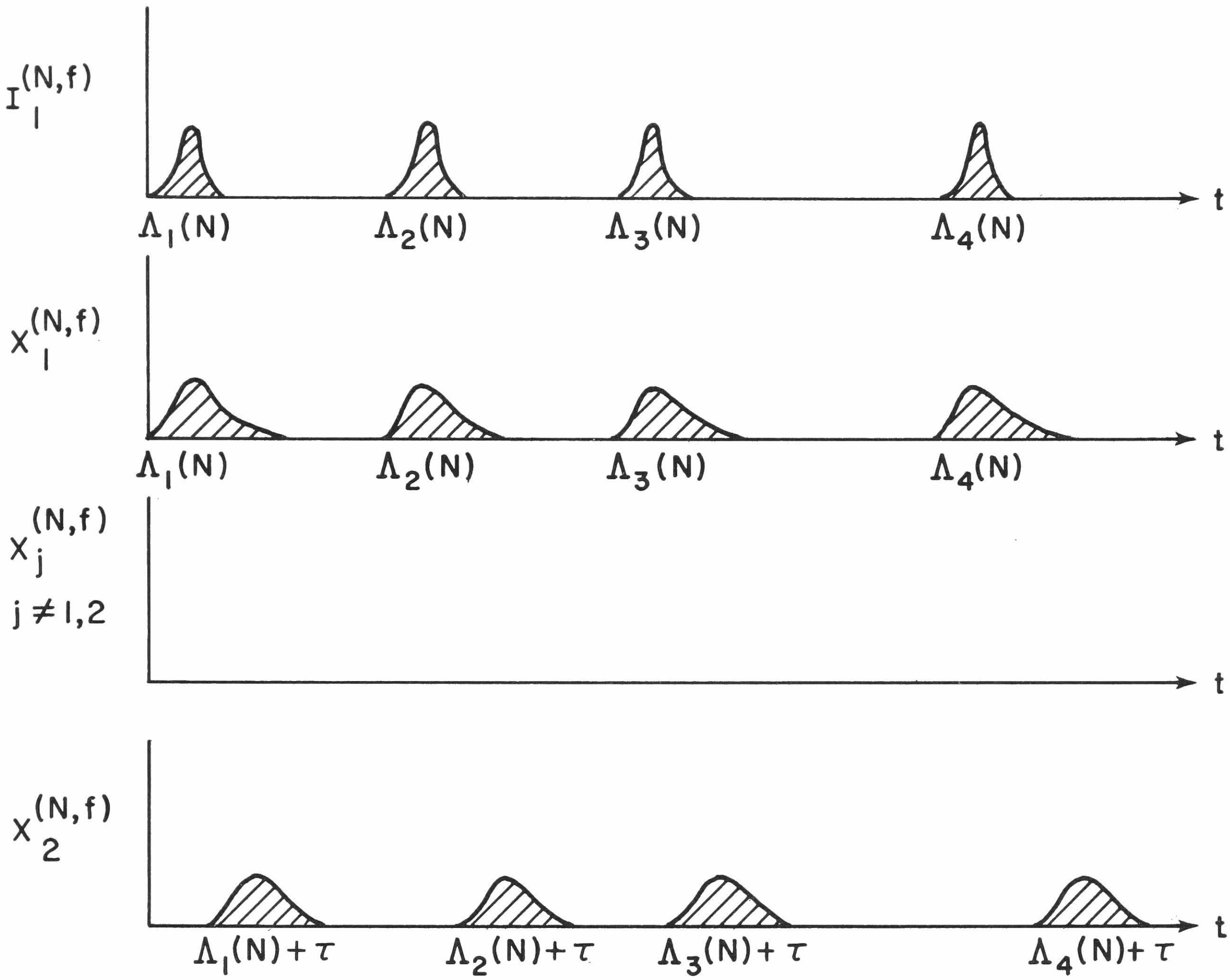
channels a later test input pulse to v_1 along e_{12} and thence to v_2 .

Suppose that the probability distribution δ_{j2} is replaced by an arbitrary probability distribution θ_j in the inputs $I_j^{(N,f)}$. Then the outputs from each v_j with $\theta_j > 0$ are affected by the input pulse f_N no matter how large N is taken. Clearly, for N taken sufficiently large, the output from v_i is approximately $\frac{\theta_i}{\theta_j}$ times as large as the output from v_j .

$$(2) \quad f_N(t) = \Sigma_{k=1}^{\infty} J(t - \Lambda_k(N)), \quad \lambda(N) << \Lambda_1(N) << \Lambda_2(N) << \ldots .$$

In this case, infinitely many input pulses occur at the source at large time separations $t = \Lambda_1(N), \Lambda_2(N), \ldots$. For this choice of f_N , we again readily conclude that the output functions $x_j^{(N,f)}$, $j \neq 1, 2$, are not affected by f_N if N is taken sufficiently large. Again the interest centers in $x_2^{(N,f)}$ for large N .

We can treat $x_2^{(N,f)}$ just as we did in Case (1) for times $t \in [0, \Lambda_2(N)]$. Treating $\Lambda_1(N)$ as the $\Lambda(N)$ of Case (1), we conclude that $x_2^{(N,f)}$ grows substantially in the interval $(\Lambda_1(N) + \tau, \Lambda_1(N) + \tau + \lambda)$ and then decays exponentially towards zero in $[\Lambda_1(N) + \tau + \lambda, \Lambda_2(N)]$. Since $\Lambda_2(N) >> \Lambda_1(N)$, we can suppose $x_2^{(N,f)}(\Lambda_2(N)) \cong 0$. Now we iterate this process. We treat $\Lambda_2(N)$ as the $\Lambda(N)$ of Case (1) and $\Lambda_1(N)$ as the $\lambda(N)$ of Case (1) . We conclude that $x_2^{(N,f)}$ grows substantially in the interval $(\Lambda_2(N) + \tau, \Lambda_2(N) + \tau + \lambda)$ and then decays exponentially towards zero in $[\Lambda_2(N) + \tau + \lambda, \Lambda_3(N)]$. This process is iterated infinitely often, and we arrive at Figure 12. The heuristic point of this example is that we can perturb the source as often as we wish with input pulses without distorting the output, just so long as the rate with which the pulses occur is sufficiently slow.



$I_1^{(N,f)}$
t
$\Lambda_1(N)$
$\Lambda_2(N)$
$\Lambda_3(N)$
$\Lambda_4(N)$
$X_1^{(N,f)}$
t
$\Lambda_1(N)$
$\Lambda_2(N)$
$\Lambda_3(N)$
$\Lambda_4(N)$
$X_j^{(N,f)}$
$j \neq 1,2$
t
$X_2^{(N,f)}$
t
$\Lambda_1(N)+\tau$
$\Lambda_2(N)+\tau$
$\Lambda_3(N)+\tau$
$\Lambda_4(N)+\tau$

Fig. 12

4D) The Nonlinear Trend in the Individual Outputs is Not Seen in the Linear Average Output

In sections (4A) - (4C), we have shown that there exists a distinctive trend in the outputs of a sequence $G^{(1,f)}$, $G^{(2,f)}$, ..., $G^{(N,f)}$, ... of outstars if, for example, we let $f_N(t) = J(t - \Lambda(N))$, where $\Lambda(N) >> \lambda(N)$. This trend is particularly evident if we let $G^{(\infty)}$ have initial data of the form $z_{1j}(0) > 0$ and $x_j(0) = \gamma$, for all $j \neq 1$, and choose $\theta_j = \delta_{j2}$. Then in every $G^{(N,f)}$, the output from each vertex of the border is the same at time $t = 0$, and we say that the output is uniformly distributed at time $t = 0$. In $G^{(1,f)}$, the distribution of outputs from the border never deviates too far from this uniform distribution since only one input pulse reaches v_1 and v_2. In $G^{(2,f)}$, the distribution of outputs from the border is slightly more peaked at v_2 for times $t \geq \Lambda(1)$. By the time we reach a $G^{(N,f)}$ for which N is very large, practically all the output from the border comes from vertex v_2 for times $t \geq \Lambda(N)$. We diagram this trend in an idealized way in Figure 13. We have set $w = \tau$ for simplicity.

We now ask how much of this striking trend is visible in the average output

$$x^{(N,f)} = \frac{1}{n} \sum_{k=1}^{n} x_k^{(N,f)}$$

of each outstar $G^{(N,f)}$, $N = 1, 2, \ldots$. We shall show that this trend need not appear at all in these averages for large times.

PROPOSITION 2.2. For any $f_N(t) = \Sigma_{k=1}^{R} J(t - \Lambda_k(N))$, where $\Lambda_{k+1}(N) - \Lambda_k(N) = \Lambda_{k+1}(M) - \Lambda_k(M) >> 0$, and R is any nonnegative integer, including ∞,

$$x^{(M,f)}(t + \Lambda_1(M)) \cong x^{(N,f)}(t + \Lambda_1(N))$$

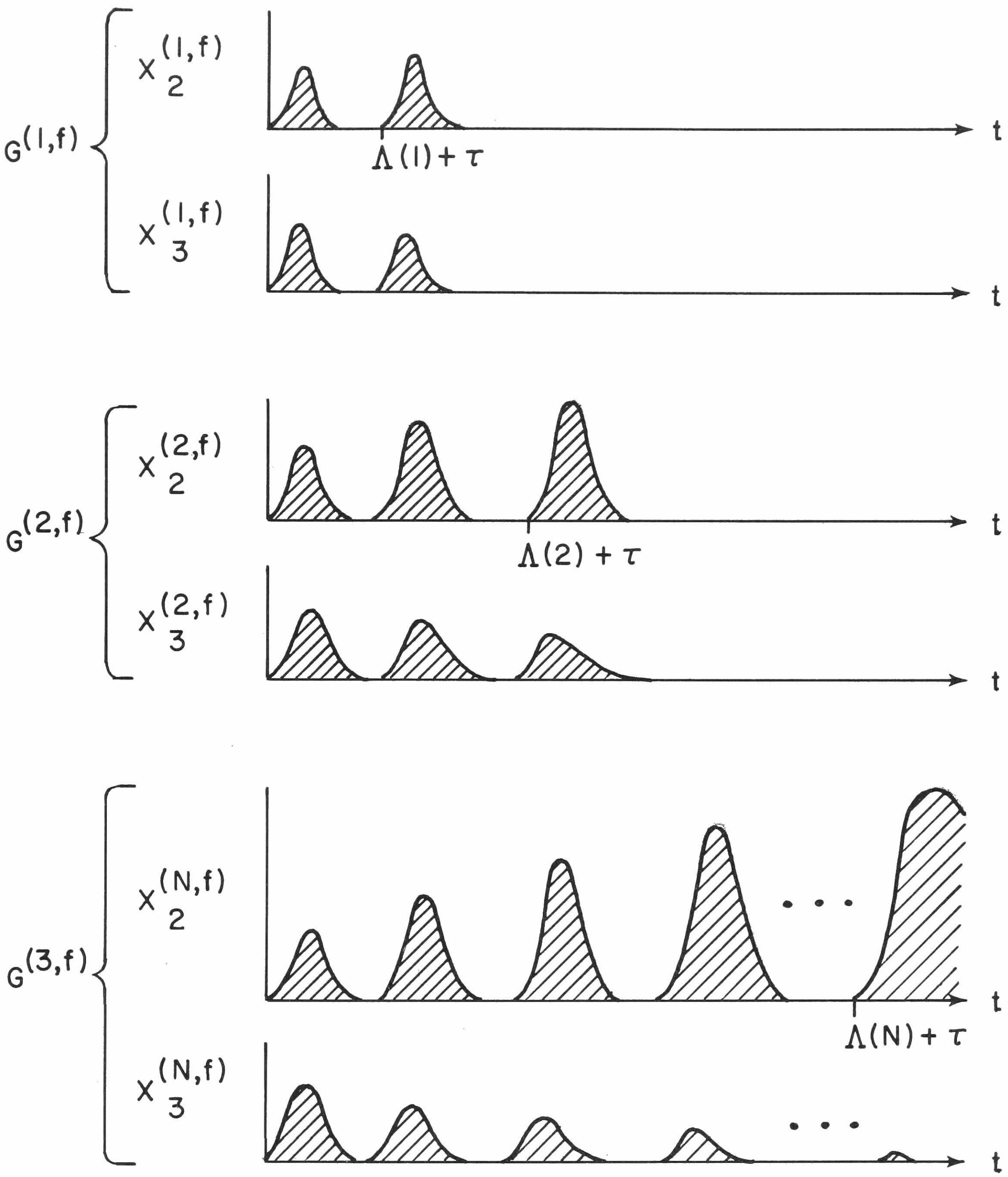

$G^{(1,f)}$
$X_2^{(1,f)}$
t
$\Lambda(1)+\tau$
$X_3^{(1,f)}$
t
$G^{(2,f)}$
$X_2^{(2,f)}$
t
$\Lambda(2)+\tau$
$X_3^{(2,f)}$
t
$G^{(3,f)}$
$X_2^{(N,f)}$
t
$\Lambda(N)+\tau$
$X_3^{(N,f)}$
t

Fig. 13

for all $t \geqslant 0$ and all M, $N = 1, 2, \ldots$.

PROOF. We prove the proposition only for the case $R = 1$. The generalization to other values of R will then be obvious. The proof relies on the fact that the average $x^{(N,f)}$ obeys a linear equation which is independent of the probability distribution $X^{(N,f)}$. This can readily be seen from (2.1) and (2.2). We omit the subscript "1" in $\Lambda_1(N)$ for simplicity.

For $t \geqslant \Lambda(N)$, summing over $j \neq 1$ in (2.2) gives

$$\left(\Sigma_{j\neq 1}\, x_j^{(N,f)}\right)^{\bullet} = -\alpha\left(\Sigma_{j\neq 1}\, x_j^{(N,f)}\right) + \beta x_1^{(N,f)}(t-\tau).$$

Adding this equation to (2.1) and dividing by n gives

$$\dot{x}^{(N,f)} = -\alpha x^{(N,f)} + \frac{\beta}{n} x_1^{(N,f)}(t-\tau) + \frac{1}{n} J(t-\Lambda(N)), \tag{2.17}$$

since by (2.1)

$$\dot{x}_1^{(N,f)} = -\alpha x_1^{(N,f)} + J(t-\Lambda(N)). \tag{2.18}$$

Integrating (2.17) gives for $t \geqslant \Lambda(N)$,

$$x^{(N,f)}(t) = x^{(N,f)}(\Lambda(N))e^{-\alpha(t-\Lambda(N))}$$
$$+ \frac{1}{n}\int_{\Lambda(N)}^{t} e^{-\alpha(t-v)}\left[\beta x_1^{(N,f)}(t-\tau) + J(v-\Lambda(N))\right] dv$$

$$\cong \frac{\beta}{n}\int_{\Lambda(N)}^{t} e^{-\alpha(t-v)} x_1^{(N,f)}(v-\tau)\,dv$$

$$+\frac{1}{n}e^{-\alpha(t-\Lambda(N))}\int_0^{t-\Lambda(N)} e^{\alpha v}J(v)\,dv. \tag{2. 19}$$

Similarly, integrating (2. 18) gives for $t \geq \Lambda(N)$

$$x_1^{(N,f)}(t) \cong e^{-\alpha(t-\Lambda(N))}\int_0^{t-\Lambda(N)} e^{\alpha v}J(v)\,dv \tag{2. 20}$$

Substituting (2. 20) into (2. 21) gives

$$x^{(N,f)}(t) \cong \frac{\beta}{n}\chi_{[\Lambda(N)+\tau,\infty)}(t)\int_0^{t-\Lambda(N)-\tau} dv\int_0^{v} e^{\alpha w}J(w)\,dw$$

$$+\frac{1}{n}e^{-\alpha(t-\Lambda(N))}\int_0^{t-\Lambda(N)} e^{\alpha v}J(v)\,dv$$

where

$$\chi_M(t) = \begin{cases} 0 & \text{if } t \notin M \\ 1 & \text{if } t \in M \end{cases}.$$

Thus for any $t \geq 0$

$$x^{(N,f)}(t+\Lambda(N)) \cong \frac{\beta}{n}\chi_{[\tau,\infty)}(t)e^{-\alpha(t-\tau)}\int_0^{t-\tau} dv\int_0^{v} e^{\alpha w}J(w)\,dw$$

$$+\frac{1}{n}e^{-\alpha t}\int_0^{t} e^{\alpha v}J(v)\,dv$$

$$\cong x^{(M,f)}(t+\lambda(M)).$$

The heuristic point of Proposition 2.2 can be stated for the case in which $R = \infty$ in the following way. Suppose that an experimentalist wants to find out how outstars work by collecting data from them. A standard rule of prudence when confronted with an unknown system is to first study the long time average output of the system. Given the outstar $G^{(N,f)}$, this average is

$$\frac{1}{t}\int_0^t x^{(N,f)}(v)\,dv,$$

where $t \gg 0$. The experimentalist will readily observe that each average output $x^{(N,f)}$ obeys a simple linear equation. He will also note that

$$\frac{1}{t}\int_0^t x^{(N,f)}(v)\,dv \cong \frac{1}{t}\int_0^t x^{(M,f)}(v)\,dv$$

for t sufficiently large and all $M, N = 1, 2, \ldots,$ since

$$x^{(N,f)}(t+\lambda(N)) \cong x^{(M,f)}(t+\lambda(M))$$

and an infinite amount of mass f_N reaches the source vertex for times $t \geq \lambda(N)$ in every $G^{(N,f)}$. A plausible inference from these data is that all the outstars $G^{(1,f)}, \ldots, G^{(N,f)}, \ldots$ obey a linear equation and that these outstars are essentially copies of one another. Both of these conclusions are totally wrong!

5. The Entropy of an Outstar

Given any probability distribution $p = (p_1, p_2, \ldots, p_{n-1})$, let the *entropy* $H(p)$ of p be defined by

$$H(p) = - \Sigma_{k=1}^{n-1} p_k \ln_2 p_k ,$$

where it is understood that $0 \ln_2 0 = 0$. This concept of entropy is familiar from information theory, and it provides a rigorous measure of the amount of information in a scheme of events ([10], [14], [17]). Using this familiar notion of entropy, we can define two kinds of entropy in any outstar $G^{(N)}$, $N = 1, 2, \ldots, \infty$. Let

$$H_X^{(N)}(t) = H(X_2^{(N)}(t), \ldots, X_n^{(N)}(t))$$

be the *vertex* (or *state*) *entropy* of the border of $G^{(N)}$ at time t. Let

$$H_y^{(N)}(t) = H(y_{12}^{(N)}(t), \ldots, y_{1n}^{(N)}(t))$$

be the *edge* (or *interaction*) *entropy* of the border $G^{(N)}$ at time t.

PROPOSITION 2. 3. Let $G^{(\infty)}$ be any outstar with an initially uniform border and input functions

$$I_1^{(\infty)} = \Sigma_{k=o}^{\infty} J(t - k(w + W))$$

and

$$I_j^{(\infty)} = \delta_{j2} \Sigma_{k=o}^{\infty} J(t - w - k(w + W)), \; j \neq 1 .$$

Then the state entropy $H_X^{(\infty)}$ and the interaction entropy $H_y^{(\infty)}$ of $G^{(\infty)}$

attain the maximum entropy $\ln_2(n - 1)$ at time $t = 0$ and approach the minimum entropy 0 as $t \to \infty$. Moreover, $H_y^{(\infty)}(t)$ decreases monotonically from maximal entropy to minimal entropy as $t \to \infty$, and

$$\lim_{N\to\infty} \lim_{t\to\infty} H_X^{(N)}(t) = \lim_{N\to\infty} \lim_{t\to\infty} H_y^{(N)}(t) = 0 \quad .$$

PROOF. The maximum entropy of $H(p)$ is $\ln_2(n - 1)$ and is attained when $p = (\frac{1}{n-1}, \ldots, \frac{1}{n-1})$. By hypothesis,

$$X_j^{(\infty)}(0) = y_{1j}^{(\infty)}(0) = \frac{1}{n-1}, \; j \neq 1 \; .$$

Thus

$$H_X^{(\infty)}(0) = H_y^{(\infty)}(0) = \ln_2(n - 1) \quad .$$

The entropy $H(p)$ is also a continuous function of p whose minimum value 0 is attained when $p = (1, 0, 0, \ldots, 0)$ (say). By Corollary 2.1, $\lim_{t\to\infty} X_j^{(\infty)}(t) = \lim_{t\to\infty} y_{1j}^{(\infty)}(t) = \delta_{j2}$. Thus

$$\lim_{t\to\infty} H_X^{(\infty)}(t) = \lim_{t\to\infty} H_y^{(\infty)}(t) = 0 \quad .$$

By Theorem 2.3, we also know that $\lim_{N\to\infty} \lim_{t\to\infty} X_j^{(N)}(t) = \lim_{N\to\infty} \lim_{t\to\infty} y_{1j}^{(N)}(t) = \delta_{j2}$.

Thus

$$\lim_{N\to\infty} \lim_{t\to\infty} H_X^{(N)}(t) = \lim_{N\to\infty} \lim_{t\to\infty} H_y^{(N)}(t) = 0 \quad .$$

To show that $H_y^{(\infty)}(t)$ decreases monotonically, note that

$$\frac{d}{dp_1} H\left(p_1, \frac{1}{n-2}(1-p_1), \frac{1}{n-2}(1-p_1), \ldots, \frac{1}{n-2}(1-p_1)\right) = \frac{\log_e\left(\frac{1-p_1}{(n-2)p_1}\right)}{\log_e (n-1)}$$

$$\begin{cases} \leq 0 & \text{if } p_1 \geq \frac{1}{n-1} \\ > 0 & \text{if } p_1 < \frac{1}{n-1} . \end{cases} \qquad (2.21)$$

Since $G^{(\infty)}$ has a uniformly distributed border at $t = 0$ and $I_j \equiv 0$, $j \neq 1, 2$,

$y_{13}^{(\infty)} \equiv y_{14}^{(\infty)} \equiv \ldots \equiv y_{1n}^{(\infty)}$ and $y_{12}^{(\infty)} = \frac{1}{n-2}(1 - y_{12}^{(\infty)})$, $j \neq 1, 2$. Thus

$$H_y^{(\infty)}(t) = H\left(y_{12}^{(\infty)}, \frac{1}{n-2}(1-y_{12}^{(\infty)}), \ldots, \frac{1}{n-2}(1-y_{12}^{(\infty)})\right).$$

Differentiating $H_y^{(\infty)}$ therefore gives

$$\dot{H}_y^{(\infty)}(t) = \frac{d}{dy_{12}} H\left(y_{12}^{(\infty)}, \frac{1}{n-2}(1-y_{12}^{(\infty)}), \ldots, \frac{1}{n-2}(1-y_{12}^{(\infty)})\right) \dot{y}_{12}^{(\infty)}$$

$$(2.22)$$

To calculate the sign of $\dot{y}_{12}^{(\infty)}$, consider Theorem 2. 2 in the light of the following facts:

(1) $y_{12}^{(\infty)}(0) - X_2^{(\infty)}(0) = 0$, (2) $y_{12}^{(2)}(0) = \frac{1}{n-1}$, and (3) $\lim_{t\to\infty} y_{12}^{(\infty)}(t) = 1$.

Thus $\dot{y}_{12}(t) \geq 0$ and $y_{12}(t) \geq \frac{1}{n-1}$. By (2. 21) and (2. 22), we therefore find that $\dot{H}_y(t) \leq 0$, or that H_y is monotone decreasing.

6. A Recursively Defined Linear Comparison System

In this section, we compare a $G^{(\infty)}$ outstar for which $\theta_j = \quad_{j2}$ and whose border is uniform at $t = 0$ with the following linear system.

$$\dot{w}_1 = -\alpha w_1 + I_1 \tag{2.23}$$

$$\dot{w}_j = -\alpha w_j + \beta \Sigma_{k=o}^{\infty} \lambda_{jk} \chi_{[n(k),\, n(k+1))}(t)\, w_1(t-\tau) + \delta_{j2} I_2(t), \quad j = 2, 3, \ldots, n \tag{2.24}$$

(*)

and

$$\dot{D}_{1j} = -uD_{1j} + \beta w_1(t-\tau)\, w_j(t) \quad , \; j = 2, 3, \ldots, n \tag{2.25}$$

where

$$I_1(t) = \Sigma_{k=o}^{\infty} J(t - k(w + W))$$

and

$$I_j(t) = \delta_{j2} \Sigma_{k=o}^{\infty} J(t - w - k(w + W)) \quad .$$

w and W are, as usual, nonnegative numbers whose sum is positive, and we write $n(k) = k(w + W)$ for simplicity. The numerical coefficients of (*) are defined by letting

$$\lambda_{jk} = \frac{D_j(k)}{\Sigma_{m=2}^{n} D_m(k)} \quad ,$$

where $D_j(k) = D_{1j}(n(k))$, $k = 0, 1, \ldots$, $j = 2, 3, \ldots, n$. (*) is linear because $w_1(t - \tau)$ in (2. 24) and (2. 25) is a known function of time by (2. 23).

The initial data of (*) are chosen to agree with that of the outstar to which it is compared. In particular, we let $w_1(v) = x_1(v) = 0$, $v \in [-\tau, 0]$, $w_j(0) = x_j(0) = 0$, and $D_{1j}(0) = z_{1j}(0) = \delta > 0$, $j \neq 1$. It is then readily shown by an iterative procedure that $w_2(t) \leq x_2(t)$ and $D_{12}(t) \leq y_{12}(t)$ for all $t \geq 0$, whereas $w_j(t) \geq x_j(t)$ and $D_{1j}(t) \geq y_{1j}(t)$ for all $t \geq 0$ and $j \neq 1, 2$. In particular, $\lambda_{2k} \leq y_{12}(n(k))$ (< 1) for all $k = 1, 2, \ldots$, and

$$\frac{w_2(t)}{\Sigma_{k=2}^{n} W_k(t)} \leq \frac{x_2(t)}{\Sigma_{k=2}^{n} x_k(t)}$$

for all $t \geq 0$. In Corollary 2. 1, we showed that $\lim_{t\to\infty} y_{12}(t) = \lim_{t\to\infty} \frac{x_2(t)}{\Sigma_{k=2}^{n} x_k(t)} = 1$ in the outstar. In the present section, we show that

$$\lim_{k\to\infty} \lambda_{2k} = \lim_{t\to\infty} \frac{w_2(t)}{\Sigma_{k=2}^{n} w_k(t)} = 1 ,$$

which provides an alternative proof of Theorem 2. 2 for outstars whose initial data are chosen as above. This alternative proof will emphasize the importance of the discrete time scale $n(k)$, $k = 1, 2, \ldots$ in determining the limiting behavior of (*). Indeed, it reduces the question of finding the limiting behavior of the nonlinear outstar to deciding the divergence of a series in the linear comparison system (*).

THEOREM 2. 4 Let (*) be given with initial data $w_1(v) = w_j(0) = 0$, $v \in [-\tau, 0]$, and $D_{1j}(0) = \delta > 0$, $j \neq 1, 2$. Then

$$\lim_{k\to\infty} \lambda_{jk} = \lim_{t\to\infty} \frac{w_j(t)}{\Sigma_{k=2}^{n} w_k(t)} = \delta_{j2}.$$

PROOF. Once $\lim_{k\to\infty} \lambda_{jk}=\delta_{j2}$ is proved, $\lim_{t\to\infty} \frac{w_j(t)}{\Sigma_{k=2}^{n} w_k(t)} = \delta_{j2}$ is easily established. The strategy for proving $\lim_{k\to\infty} \lambda_{jk} = \delta_{j2}$ is given below. The details are in Appendix A. First we show how a series enters the proof.

By (2.24),(2.25), and the uniform distribution of initial data, it is readily seen that $D_{13}(t)=D_{14}(t)=\ldots=D_{1n}(t)$. Since also $\Sigma_{m=2}^{n}\lambda_{mk}=1$, we conclude that $\lambda_{2k}=1-(n-2)\lambda_{3k}$. Thus $\lim_{k\to\infty} \lambda_{2k}=1$ iff $\lim_{k\to\infty} \lambda_{2k}/\lambda_{3k}=\infty$. Since $\lambda_{2k}/\lambda_{3k}=D_2(k)/D_3(k)$, it suffices to show that $\lim_{k\to\infty} D_2(k)/D_3(k)=\infty$ to prove the theorem. The main step in doing this is to show that $D_2(k)/D_3(k)$ has the following series representation

$$\frac{D_2(k)}{D_3(k)} = 1+\frac{1}{D_{12}(0)}\sum_{i=1}^{k} \Phi_i R^{-(i-1)} \prod_{j=1}^{i} \left(\frac{V_{j-1} + c}{V_{j-1} + c + R^{-(j-1)} \Phi_j} \right), \tag{2.26}$$

where $R=e^{-u(w+W)}$,

$c=(n-1)D_{12}(0)$,

$$\Phi_k = \beta \int_0^{n(1)} e^{(u-\alpha)v} w_1(v-\tau+n(k-1)) \cdot$$

$$\left[w_2(n(k-1))+\int_0^{v} e^{\alpha u} I_2(u+n(k-1))\,du\right] dv, \tag{2.27}$$

$$\Psi_k = \beta^2 \int_0^{n(1)} e^{(u-\alpha)v} w_1(v-\tau+n(k-1)) \cdot$$
$$\int_0^{v} e^{\alpha u} w_1(u-\tau+n(k-1))\,du\,dv,$$

(2.28)

and $V_k = \Sigma_{j=1}^{k} R^{1-j}(\Phi_j + \Psi_j)$. Once this is shown it will remain to show only that the series

$$T = \Sigma_{i=1}^{\infty} \Phi_i R^{-(i-1)} \Pi_{j=1}^{i} \left(\frac{V_{j-1} + c}{V_{j-1} + c + R^{-(j-1)} \Psi_j} \right)$$

diverges.

(2.26) is derived by finding sufficiently many recursions involving the quantities $D_2(k), D_3(k)$, and know functions of I_1 and initial data. The proof of these recursions depends on the fact that only the coefficient λ_{jk} appears in (2.24) when $t \in [n(k), n(k+1))$. This is the first appearance of the discrete time scale $n(k), k=1,2,...$ First we find a recursion expressing $\frac{D_2(k)}{D_3(k)} - \frac{D_2(k-1)}{D_3(k-1)}$ in terms of $D_3(k-1)$, $D_2(k-1)+((n-2)D_3(k-1)$, and known functions. Next we show that $D_2(k-1)+(n-2)D_3(k-1)$ obeys a recursion which can be solved in terms of known functions. It is not surprising that this is possible, because $D_2(k-1)+(n-2)D_3(k-1)$ is the analog of the sum $z^{(1)} = \Sigma_{k=2}^{n} z_{1k}$ in an outstar, and this sum obeys equation (2.6), which is independent of the unknown probabilities X_j and y_{1j}. $D_3(k-1)$ can then be disposed of in a similar way. This allows us to write $\frac{D_2(k)}{D_3(k)} - \frac{D_2(k-1)}{D_3(k-1)}$ as a known function of I_1 and initial data. A simple recursion on k then gives the desired series representation (2.26).

The expression for $\frac{D_2(k)}{D_3(k)} - \frac{D_2(k-1)}{D_3(k-1)}$ is given by

$$\frac{D_2(k)}{D_3(k)} - \frac{D_2(k-1)}{D_3(k-1)} = \frac{\Phi_k\left(D_2(k-1)+(n-2)D_3(k-1)\right)}{D_3(k-1)\left(\Psi_k + D_2(k-1)+(n-2)D_3(k-1)\right)},$$

where $D_2(k)+(n-2)D_3(k)=R^k(V_k + c)$

and $$\frac{1}{D_3(k)} = \frac{R^{-k}}{D_{12}(o)} \prod_{j=1}^{k}\left(\frac{V_{j-1} + c}{V_{j-1} + c + \Psi_j R^{-(j-1)}}\right)$$

(See Appendix A.) Thus

$$\frac{D_2(k)}{D_3(k)} - \frac{D_2(k-1)}{D_3(k-1)} = \frac{R^{-(k-1)}}{D_{12}(0)} \Phi_k \prod_{j=1}^{k}\left(\frac{V_{j-1} + c}{V_{j-1} + c + \Psi_j R^{-(j-1)}}\right),$$

which when iterated gives the series (2.27) for $D_2(k)/D_3(k)$. This completes the first part of the proof.

The second part of the proof is to establish that the series T diverges. Here the discrete time scale $n(k), k=1,2,\ldots$ makes a second, and crucial, appearance. Namely, it allows us to conclude that the sequences $\Phi_1, \Phi_2, \ldots$ and $\Psi_1, \Psi_2, \ldots$ are monotone increasing. These sequences are also bounded because the inputs I_1 and I_2 are bounded. From these two facts follows immediately the existence of the finite limits $\Phi = \lim_{i\to\infty} \Phi_i$ and $\Psi = \lim_{i\to\infty} \Psi_i$. Once we know that the limits Φ and Ψ exist, we can apply the ratio test to the series T to test for its divergence.

In the course of proving that the sequence $\Phi_1, \Phi_2, \ldots$ is monotonic, we need to know that all the terms in the series (2.27) are nonnegative, since then $\frac{\lambda_{2k}}{\lambda_{3k}} \left(= \frac{D_2(k)}{D_3(k)}\right)$ and hence λ_{2k} is monotone increasing in k. In the analogous outstar, nonnegativity guarantees that $y_{12}(t)$ is monotone increasing.

Nonnegativity also guarantees that Φ and Ψ are positive.

Given the existence of the positive limits Φ and Ψ, the ratio test implies that T will diverge if

$$\lim_{i\to\infty} \frac{\Phi_i S}{\Phi_{i-1}} \frac{V_{i-1} + c}{V_{i-1} + c + \Phi_i S^{i-1}} > 1,$$

where $S=R^{-1}=e^{\alpha(W+W)} > 1$; that is, if

$$\lim_{i\to\infty} \frac{V_{i-1} + c}{V_{i-1} + c + \Phi_i S^{i-1}} > \frac{1}{S} .$$

Once this is established, $\lim_{k\to\infty} \lambda_{jk}=\delta_{j2}$ is immediate. (See Appendix A.)

PREDICTION THEORETIC INTERPRETATION

PART II

1. Introduction

We now give the results of Part I a prediction theoretic interpretation. Our goal is to construct laws for a machine $\mathcal{M}$ which can be taught to predict the event B whenever the event A occurs. This goal can be stated in several related ways. We can say that we wish to teach the machine that the transition $A \rightarrow B$ is correct, or that we wish to teach the machine the list AB. Phrased in this way, our task can be described by analogy with the task of teaching lists of letter to an idealized human subject, who shall henceforth be denoted by $\mathcal{S}$. Suppose that we wish to teach $\mathcal{S}$ the list of letters AB. A standard way of doing this is to repeat the list AB to $\mathcal{S}$ several times. To find out if $\mathcal{S}$ has learned the list as a result of these list interpretations, the letter A alone is then said to $\mathcal{S}$. If $\mathcal{S}$ responds by saying the letter B in return, and $\mathcal{S}$ does this whenever A alone is said, then we have good evidence that $\mathcal{S}$ has indeed learned the list AB. Thus $\mathcal{S}$ learns to predict the event B whenever the event A occurs as a result of repeated presentations of the list AB.

In this section, we suggest one way of translating this intuitive idea of learning into formal terms. We can easily think of several desirable properties which a machine that learns a list of events in this way might profitably have. We state these properties here in a somewhat colorful language to aid the reader in comparing and contrasting his intuitive concepts of learning with the particular formal translation table that we shall set down for these properties. The translation table that we shall provide is a very special one, to be sure,

since it is intended to deal with the particularly simple case of an outstar.

1) Practice Makes Perfect.

The more often the list AB of events is repeated to the machine $\mathcal{M}$, the better becomes $\mathcal{M}$'s prediction of B given A. Moreover, if the list AB is repeated indefinitely often, then $\mathcal{M}$'s prediction of B given A comes as close as we wish to a perfect prediction.

2) An Isolated System Suffers No Memory Loss.

If we succeed in teaching the list AB to $\mathcal{M}$ to a given degree of accuracy, then $\mathcal{M}$ remembers the list with approximately this accuracy just so long as no new teaching occurs.

3) An Isolated System Remembers without Continually Practicing.

In everyday life, it is a commonplace experience that facts can be remembered for a substantial time in the absence of continual overt practice. We shall construct a machine that also has a good memory even when it does not practice. Indeed, its memory sometimes spontaneously improves even without practice (i. e. "reminiscence" occurs; [16], p. 509).

4) The Act of Making a Correct Prediction Can Reoccur Indefinitely Often without Retraining.

Suppose that $\mathcal{M}$ knows the list AB of events. It would be most unpleasant if the very act of predicting B, given A, erased the record within $\mathcal{M}$ that B is indeed the correct reply to A. If this were true, we would have to reteach the list AB every time a correct prediction occurred. In the present system, the act of recall can occur as many times as we please without requiring the retraining of $\mathcal{M}$.

Properties (1)-(4) show that once a list AB of events is taught to the machine $\mathcal{M}$, retention of the list is quite stable. The next property shows that this stability does not prevent $\mathcal{M}$ from adapting to new experiences.

5) <u>All Errors Can Be Corrected.</u>

Suppose after $\mathcal{M}$ learns the list AB it is found that really the event C should follow the event A. Then B is , by fiat, an <u>error</u> whenever it follows A. We shall see that this error can always be corrected if $\mathcal{M}$ then practices the list AC sufficiently often.

We now make properties (1)-(5) rigorous by translating them into theorems of Part I about outstars.

2. The Machine.

The machine $\mathcal{M}$ which we construct here obeys the equations (2.1)-(2.4) of an outstar. Once this machine is understood, the same basic concepts can be applied toasystem given by any semistochastic matrix P, as in Chapter 1. $\mathcal{M}$ consists of n states, namely the n vertices v_i of the outstar, and these states interact with one another along the directed edges e_{1i}. The machine $\mathcal{M}$ is manipulated by an experimenter $\mathcal{E}$ whose goal is to teach $\mathcal{M}$ to predict the event B given the event A. The experimental manipulations created by $\mathcal{E}$ are represented by the input vector function $C=(I_1, I_2, \ldots, I_n)$. The outputs which these manipulations produce are represented by the output vector function $X=(x_1, x_2, \ldots, x_n)$. In particular, the input function I_j represents the total history of experimental manipulations performed on state v_j.

Suppose for example that

$$I_1 = \Sigma_{k=o}^{\infty} J_1(t-k(w+W)$$

and

$$I_j = \theta_j \Sigma_{k=o}^{\infty} J_2(t-w-k(w+W)), \quad j\neq 1.$$

Then $\mathcal{M}$ is an outstar of type $G^{(\infty)}$, as treated in Corollary 2.1. Each function

$J_1(t-k(w+W))$ and $J_2(t-w-k(w+W))$ signifies the occurrence of an experimentally created event. $J_1(t-k(w+W))$ is an event which begins at the source vertex v_1 at time $t=k(w+W)$ and lasts until time $t=k(w+W)+\lambda_1$. The function $I_1=\Sigma_{k=o}^{\infty} J_1(t-k(w+W))$ signifies the occurrence at the source v_1 of a periodic succession of identical events with the waveform $J_1(t)$ at the times $t=0,\ w+W,\ 2(w+W),\ldots$ These events each last λ_1 time units. Similarly, the function $I_j=\theta_j\Sigma_{k=o}^{\infty} J_2(t-w-k(w+W))$ signifies the occurrence at the border vertex V_j of a periodic succession of identical events with waveform $\theta_j J_2$ at the times $t=w,\ 2w+W,\ 3w+2W,\ldots$. These events each last λ_2 time units. Every vertex v_j of the border receives a periodic succession of events whose waveform is identical except for the weights θ_j. That is, the experimenter distributes a fraction θ_j of the waveform J_2 to each vertex V_j of the border at periodic intervals. Corollary 2.1 assures the experimenter that his unceasing labors do not go unnoticed by the outstar. Indeed the normalized vertex functions X_j and the normalized edge functions y_{1j} are both responsive to the experimenters's choice of weights θ_j and gradually adopt these weights as their own no matter what their weights were initially.

3. Repeating the List AB N Times.

Now that we know what an event means in an outstar, it is simple to translate into formal terms the intuitive idea of presenting a list AB of events to the outstar N times. Firstly we must assign a state of the outstar to each symbol of an event. If for example we are given twenty-six symbols A, B, C, . . . , Z, then we assign v_1 to A, v_2 to B, v_3 to C, and so on down to v_{26} and Z. Given this assignment of symbols to states, suppose that an experimenter wishes to teach an outstar to predict B given A. He must indicate to the outstar in some way that B is the "correct" successor of A. He does this by repeating the desired sequence AB several times. The only way to say a sequence AB to an outstar is to create perturbations at the vertices V_1 and V_2 which stand for A and B, respectively. Thus

one occurrence of the sequence AB is translated into an outstar's mechanism by the arrival of an input pulse $J(t-k(w+W))$ at v_1 and of an input pulse $J(t-w-k(w+W))$ at v_2 w time units later. N periodic representations of the sequence AB, starting at time zero, is translated into an outstar's mechanism as an input function $I_1^{(N)}(t)=\Sigma_{k=o}^{N-1} J(t-k(w+W))$ for vertex v_1, an input function $I_2^{(N)}(t)=\Sigma_{k=o}^{N-1} J(t-w-k(w+W))$ for vertex v_2, and input functions $I_j^{(N)}\equiv 0$ for all other vertices $j\neq 1,2$; that is, as an N-truncation $G^{(N)}$. $G^{(N)}$ is thus the outstar which results when AB is repeated N times at a fixed rate by the experimenter.

To test whether or not $G^{(N)}$ has learned to predict B given A, the experimenter presents A to $G^{(N)}$ at a later time and sees whether or not $G^{(N)}$ knows that B is the correct prediction. That is, the experimenter creates an input pulse at v_1 and waits to see if the output created in this way comes only from v_2. As soon as A occurs, however, the outstar is no longer of type $G^{(N)}$.

Suppose, for example, that A occurs in $G^{(N)}$ at time $t=\Lambda(N)$. This means that the input pulse $f_N(t)=J(t-\Lambda(N))$ occurs at the source v_1. The total input to the source is therefore $\Sigma_{k=o}^{N-1} J(t-k(w+W))+f_N(t)$. This is the input of an outstar of type $G^{(N,f)}$. An outstar of type $G^{(N,f)}$, where $f_N(t)=J(t-\Lambda(N))$, is thus a machine subjected to N presentations of the list AB followed by a single presentation of A on a test trial.

4. "Practice Makes Perfect".

Suppose now that a machine of type $G^{(N,f)}$ is given. That is, the experimenter has presented AB to the machine N times and then presents A alone. The experimenter wants the machine to predict B after A occurs. This means that the output from the border created by f_N ought to come only from v_2 if the machine knows the list AB. Theorem 2.3f shows that the output comes increasingly from v_2 as N increases. This means that the machine learns to predict B given A with ever greater precision as it

receives ever more trials on which to practice the sequence AB. In this sense, the outputs at large times from the sequence $G^{(1,f)}, G^{(2,f)}, \ldots, G^{(N,f)}, \ldots$ of outstars, where $f_N(t)=J(t-\Lambda(N))$, exemplify the proverb "practice makes perfect" in our formal translation table.

This proverb is the first property stated in Section 1. The second property is that "an isolated system suffers no memory loss". An "isolated system" is manifestly one that is input-free. Property 2 can thus be stated formally as follows. The probability distributions $X_j^{(N,f)}$ and $y_{1j}^{(N,f)}$ of an outstar of type $G^{(N,f)}$ remain essentially fixed for all large t. This is proved in Theorem 2. 3f. The third property is that "an isolated system remembers without continually practicing". This is the statement of Section 5 that the probability distributions $X_j^{(N)}$ and $y_j^{(N)}$ remain fixed even as the outputs $x_j^{(N)}$ decay exponentially to zero. A similar remark holds in an outstar of type $G^{(N,f)}$. Consider an experimenter who is studying $G^{(N,f)}$ for times $t\in[\lambda(N),\Lambda(N)]$; that is, after AB has occurred N times and before A alone occurs. He will certainly observe the rapid exponential decay of all the outputs $x_j^{(N,f)}$ and might well therefore be led to conclude that the effects of saying AB to the outstar N times wear off rapidly. The outstar provides no overt evidence (e. g., no "overt practice") to the experimenter during this time that any record whatever of his having presented AB endures. Nonetheless, by Theorem 2. 3f, shortly after A is presented to the outstar at time $t=\Lambda(N)$, the output is produced by B alone if AB has been said sufficiently often in the past. Memory sometimes spontaneously improves without practice because $\dot{y}_2^{(N,f)}(t)\geq 0$ for $N\gg 1$ and all sufficiently large t.

The fifth property of Section 1 is that "all errors can be corrected". This property is discussed in the next section.

5. Error Correction and Global Theorems.

Suppose that an experimenter has taught an outstar the list AB by presenting this list N times to the outstar at a fixed rate. If N is taken

sufficiently large, Theorem 2.3 guarantees that the list can be learned to an arbitrary degree of accuracy. After accomplishing this goal, suppose that another experimenter comes upon the outstar. This experimenter wishes to teach the outstar the list AC. He tests whether the outstar already knows this list by presenting the test pulse $\tilde{J}_1(t-\Lambda_1)$ to vertex v_1 at time $t=\Lambda_1$. The output created in this way comes almost exclusively from B. Because this experimenter wants C to be the output instead of B, he interprets the output from B as an error. To correct this error, he begins at time $t=\Lambda_2$ to present the list AC to the outstar M times at a fixed rate of speed, where M is chosen sufficiently large to offset the previous N occurrences of the "incorrect" list AB.

The input history of this outstar can be written as

$$I_1(t) = \sum_{k=0}^{N-1} J_1(t-k(w_1+W_1)) + \tilde{J}_1(t-\Lambda_1) + \sum_{k=0}^{M-1} \tilde{J}_1(t-\Lambda_2-k(w_2+W_2)),$$

$$I_2(t) = \sum_{k=0}^{N-1} J_2(t-w_1-k(w_1+W_1)),$$

$$I_3(t) = \sum_{k=0}^{M-1} \tilde{J}_3(t-\Lambda_2-w_2-k(w_2+W_2)),$$

and $I_j(t) \equiv 0, \quad j \neq 1,2,3,$

where J_i and $\tilde{J}_i$ are input pulses that are positive in $(0, \lambda_i)$ and $(0, \tilde{\lambda}_i)$, respectively, $i=1,2$, and w_i and W_i are positive numbers, $i=1,2$. The basic question is: by repeating AC a sufficiently large numbers of times M, can the record of previous occurrences of AB N times be erased from within the outstar? The answer is yes. This is because of two facts: (1) the outstar

has positive data at time $t = \Lambda_1 + \tilde{\lambda}_1$ and in the interval $[\Lambda_1 + \tilde{\lambda}_1, \infty)$ the inputs are

$$I_1(t) = \sum_{k=0}^{M-1} \tilde{J}_1(t - \Lambda_2 - k(w_2 + W_2)),$$

$$I_3(t) = \sum_{k=0}^{M-1} \tilde{J}_3(t - \Lambda_2 - w_2 - k(w_2 + W_2)),$$

and

$$I_j(t) \equiv 0, \quad j \neq 1, 3.$$

(2) Theorem 2.3 is true for any outstar with positive data and inputs of the form given in (1).

Thus the possibility of correcting all errors in an outstar depends on two facts: (1) invariance of the set of all initial data under inputs: no matter what inputs occur in a finite time interval, just so long as they are nonnegative the functions of the outstar will remain positive; and (2) the limit of ratios of solutions is independent of the initial data: Theorem 2.3f is true no matter what the initial data is, just so long as it is positive.

This discussion completes our formal translation table of the heuristic properties (1)-(5) of Section 1 in the case of an outstar. We remark in passing that properties (1)-(5) do not hold for systems characterized by arbitrary semistochastic matrices P. The way in which a system learns to predict depends in an essential way on the matrix P that characterizes it; i.e., on its "geometry". For example, in Chapter 3 we shall study a system which forgets everything it has ever learned if it does not practice. This system has the matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

and the coefficient graph of Figure 14

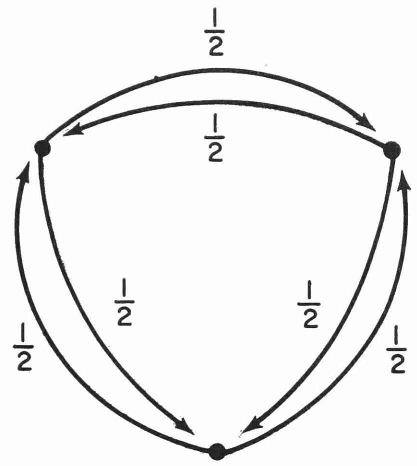


Fig. 14

By contrast, the system characterized by the closely related coefficient matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and the coefficient graph of Figure 15

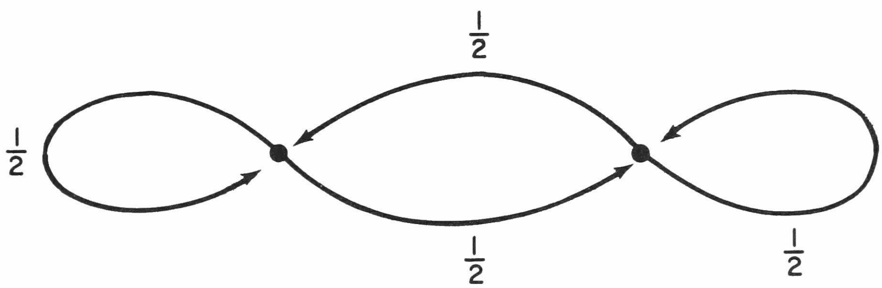


Fig. 15

does not forget if it does not practice.

6. Linear and Statistical Prediction.

The deterministic predictions of an outstar are described in terms of a nonlinear system with a continuous time scale in which events last throughout a whole time interval. A more familiar kind of prediction is statistical prediction using a linear system with a discrete time scale in which events are instantaneous. Such systems occur in contemporary mathematical learning theory ([1], [5]). A typical variable in learning theory is, for example, the probability $P_{AB}(k)$ that B follows A after k prior repetitions of AB.

The linear system of Section 6, Part I, can also be interpreted as a statistical learning theory. Simply let $P_{AB}(k)=\lambda_{2k}$, $P_{AC}(k)=\lambda_{3k}$, and so on, and think of these discrete probabilities as being embedded in the continuous system of Section 6. Then $\lim_{k\to\infty}\lambda_{2k}=1$ reads "the probability of B given A increases to 1 if AB is practiced sufficiently often". Every outstar gives rise to a linear comparison system in an obvious way. The analog of $\lim_{t\to\infty} y_{1j}(t)=\theta_j$ in Theorem 2.2 should be $\lim_{k\to\infty} P_{A\cdot}(k)=\theta_{\cdot}$, which is the statement of "probability matching".

CHAPTER III

GLOBAL RATIO LIMIT THEOREMS FOR COMPLETE GRAPHS

1. AN INPUT-FREE GRAPH WITHOUT LOOPS

In this section, we consider a system which exhibits far more nonlinear feedback than did the outstars of Chapter II. This system has the coefficient matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

and a zero lag time ($\tau = 0$). It therefore obeys the equations

$$\begin{aligned} \dot{x}_i(t) &= -\alpha x_i(t) + \beta \textstyle\sum_{k=1}^{3} x_k(t) y_{ki}(t) + I_i(t), \quad i = 1,2,3, \\ y_{ki}(t) &= \frac{z_{ki}(t)}{\sum_{j=1}^{3} z_{kj}(t)}, \quad k,i = 1,2,3, \qquad (*) \\ \dot{z}_{ki}(t) &= -u z_{ki}(t) + \beta x_k(t) x_i(t), \quad k \neq i, \\ z_{ii}(t) &= 0. \end{aligned}$$

The coefficient graph for (*) is given in Figure 14. (*) is called a <u>complete graph</u> because every vertex is connected by a pair of directed edges to every other vertex. Since no edges of the form shown in Figure 16 occur, (*) is called a complete graph <u>without</u> <u>loops</u>.

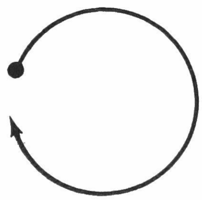


Fig. 16

We shall study global ratio limit theorems for (*) just as we did in the previous chapter for outstars. The results for the complete graph are quite different from the results for outstars in at least one important respect. In the case of outstars, the coefficient matrix $P = \|p_{ij}\|$ does not uniquely determine the limits of ratios when the outstar is input-free. Indeed, whenever $y_{1j}(0) = X_j(0)$ in an input-free outstar, we concluded in Theorem 2.1 that $y_{1j}(t) = X_j(t) =$ constant. Arbitrary probability distributions can therefore occur as limits $\lim_{t\to\infty} y_{1j}(t)$ in an input-free outstar. In the case of complete graphs such as (*), the coefficient matrix P exerts a much stronger grip on the possible limits $\lim_{t\to\infty} y_{ij}(t)$. Indeed, we now consider a case of special interest for which the coefficient matrix uniquely determines these limits and, moreover, $\lim_{t\to\infty} y_{ij}(t) = p_{ij} = \frac{1}{2}(1 - \delta_{ij})$. This fact can be interpreted, in the manner of Chapter I, Part II, by saying that the system "forgets" if it does not "practice". In Section 4, we shall find that the complete graph <u>with</u> loops does not forget if it does not practice.

In order to discuss the complete graph without loops, we cannot work directly with (*). Just as in the case of outstars, we must work instead with various probability distributions which are associated with (*). These distributions are $X_i = \frac{x_i}{x}$, $y_{jk} = \frac{z_{jk}}{z^{(j)}}$, and $x_{jk} = \frac{x_k}{x^{(j)}}$, where $x = \sum_{k=1}^{3} x_k$, $z^{(j)} = z_{ji} + z_{jk}$, $x^{(j)} = x_i + x_k$, and the indices are chosen so that $\{i, j, k\} = \{1, 2, 3\}$. (*) can be expressed in terms of these distributions as follows in the input-free case.

<u>LEMMA 3.1.</u> Let (*) be input-free with arbitrary positive initial data. Then the probability distributions X_i and y_{jk} satisfy the equations

$$\dot{X}_i = \beta(-X_i + X_j y_{ji} + X_k y_{ki}), \{i, j, k\} = \{1, 2, 3\}, \quad (3.1)$$

and

$$\dot{y}_{jk} = G_j\left(\frac{X_k}{1 - X_j} - y_{jk}\right), \quad j \neq k \quad (3.2)$$

where

$$G_j = \frac{\beta x_1 x^{(1)}}{z^{(1)}} = \frac{d}{dt} \log\left(\gamma_j + \int_0^t e^{\sigma v} X_j (1 - X_j)\, dv\right), \tag{3.3}$$

and

$$\gamma_j = \frac{z^{(j)}(0)}{\beta x^2(0)} > 0 .$$

<u>PROOF</u>: The equation for X_i is derived as follows.

$$\dot{X}_i = \frac{1}{x}\left(\dot{x}_i - x_i \frac{\dot{x}}{x}\right)$$

where

$$\dot{x}_i = -\alpha x_i + \beta(x_j y_{ji} + x_k y_{ki})$$

and

$$\dot{x} = (\beta - \alpha) x .$$

Thus

$$\dot{X}_i = \beta(-X_i + X_j y_{ji} + X_k y_{ki}) . \tag{3.1}$$

The equation for y_{jk} has the following derivation. By hypothesis,

$$\begin{aligned} \dot{z}_{jk} &= -u z_{jk} + \beta x_j x_k \\ &= -u z_{jk} + \beta x^2 X_j X_k . \end{aligned}$$

Since

$$z^{(j)} = z_{ji} + z_{jk} ,$$

$$\begin{aligned} \dot{z}^{(j)} &= -u z^{(j)} + \beta x^2 X_j (X_i + X_k) \\ &= -u z^{(j)} + \beta x^2 X_j (1 - X_j). \end{aligned}$$

In integral form this equation is

$$z^{(j)}(t) = e^{-ut}\left(z^{(j)}(0) + \beta\int_0^t e^{uv}x^2 X_j(1-X_j)\,dv\right), \quad t \geqslant 0.$$

Using the fact that

$$\dot{x} = (\beta - \alpha)x,$$

which in integral form can be expressed as

$$x(t) = x(0)\,e^{(\beta-\alpha)t}, \qquad t \geqslant 0,$$

this equation becomes

$$z^{(j)}(t) = e^{-ut}\left(z^{(j)}(0) + \beta x^2(0)\int_0^t e^{\sigma v} X_j(1-X_j)\,dv\right), \tag{3.4}$$

where $\sigma = u + 2(\beta - \alpha)$.

Since $y_{jk} = z_{jk}/z^{(j)}$,

$$\dot{y}_{jk} = \frac{1}{z^{(j)}}\left(\dot{z}_{jk} - z_{jk}\frac{\dot{z}^{(j)}}{z^{(j)}}\right)$$

$$= \frac{1}{z^{(j)}}\left[-u z_{jk} + \beta x^2 X_j X_k - z_{jk}\left(-u + \frac{\beta x^2 X_j(1-X_j)}{z^{(j)}}\right)\right]$$

$$= \frac{\beta x^2 X_j}{z^{(j)}} \left[X_k - y_{jk}(1 - X_j) \right]$$

$$= \frac{\beta x^2(0) e^{2(\beta-\alpha)t} X_j (1 - X_j)}{z^{(j)}} \left(\frac{X_k}{1 - X_j} - y_{jk} \right).$$

Letting

$$G_j = \frac{\beta x^2(0) e^{2(\beta-\alpha)t} X_j (1 - X_j)}{z^{(j)}},$$

we find

$$\dot{y}_{jk} = G_j \left(\frac{X_k}{1 - X_j} - y_{jk} \right). \tag{3.2}$$

(3.4) can be used to express G_j in a convenient form as follows.

$$G_j = \frac{\beta x^2(0) e^{\sigma t} X_j (1 - X_j)}{z^{(j)}(0) + \beta x^2(0) \int_0^t e^{\sigma v} X_j (1 - X_j) dv}$$

$$= \frac{e^{\sigma t} X_j (1 - X_j)}{\gamma_j + \int_0^t e^{\sigma v} X_j (1 - X_j) dv}$$

$$= \frac{d}{dt} \log \left(\gamma_j + \int_0^t e^{\sigma v} X_j (1 - X_j) dv \right), \tag{3.3}$$

where $\gamma_j = \dfrac{z^{(j)}(0)}{\beta x^2(0)} > 0.$

Equations (3.1), (3.2), and (3.3) are used as the starting point for proving the following theorem. This theorem provides three kinds of infor-

mation about (*): its limiting behavior as $t \to \infty$, a summary of its possible oscillations in $[0, \infty)$, and conditions on the coefficients α, β, and u under which the limits are unique for a large set of initial data.

THEOREM 3.1. Let (*) be input-free and suppose the numerical coefficients α, β, and u are arbitrary positive numbers. Then for any positive initial data satisfying $z_{ij}(0) = z_{ji}(0)$, $i, j = 1, 2, 3$, the following conclusions hold:

(1) (limiting behavior) All the ratios X_i and y_{jk} have limits $Q_i = \lim_{t\to\infty} X_i(t)$ and $P_{jk} = \lim_{t\to\infty} y_{jk}(t)$ as $t \to \infty$, which satisfy the system of equations

$$\frac{1}{2} \geq Q_i = Q_j P_{ji} + Q_k P_{ki}, \quad \{i, j, k\} = \{1, 2, 3\}.$$

In particular,

$$\lim_{t\to\infty} x_i(t)\, e^{(\alpha-\beta)t} = Q_i \sum_{k=1}^{3} x_k(0).$$

(2) (oscillations) For all indices $\{i, j, k\} = \{1, 2, 3\}$, the functions $f_{ij} = x_i - x_j$, $g_{ijk} = z_{ij} - z_{ik}$, $h_{ijk} = x_i z_{jk} - x_k z_{ji}$, and $\dot{y}_{ij}$ change sign at most once. f_{ij} and g_{kij} do not change sign at all if $f_{ij}(0) g_{kij}(0) \geq 0$, while h_{kij} and $\dot{y}_{ij}$ do not change sign at all if $f_{ij}(0)\, g_{kij}(0) \geq 0$ and $h_{kij}(0) \geq 0$. Moreover, $f_{ij}(0)\, g_{kij}(0) \geq 0$ implies $f_{ij}(t)\, g_{kij}(t) \geq 0$ for all $t \geq 0$, while $f_{ij}(0)\, g_{kij}(0) > 0$ and $h_{kij}(0)\, g_{kij}(0) > 0$ imply $f_{ij}(t)\, g_{kij}(t) > 0$ and $h_{kij}(0)\, g_{kij}(t) > 0$ for all $t \geq 0$.

(3) (uniqueness) If, moreover, the coefficients satisfy the inequality $\sigma \equiv u + 2(\beta - \alpha) > 0$, then $Q_i = \frac{1}{3}$, $i = 1, 2, 3$ and $P_{jk} = p_{jk} = \frac{1}{2}(1 - \delta_{jk})$, $j, k = 1, 2, 3$.

The following remarks help to visualize the geometrical meaning of this theorem, say for the case $\sigma > 0$ in which uniqueness of limits holds.

(a) (2) shows, for example, that if $x_i(0) > x_j(0)$ and $z_{ki}(0) > z_{kj}(0)$,

then $x_i(t) > x_j(t)$ and $z_{ki}(t) > z_{kj}(t)$ for all $t \geq 0$. Thus if a common ordering occurs in corresponding edges and vertices at $t = 0$, then this ordering propagates through time (i.e., it is a "geometrical" property of the graph). $z_{ki}(t) > z_{kj}(t)$ is equivalent to $y_{ki}(t) > \frac{1}{2}$. Since $\dot{y}_{ki}$ changes sign at most once, y_{ki} has a graph of either the form (A) or (B) given in Figure 17.

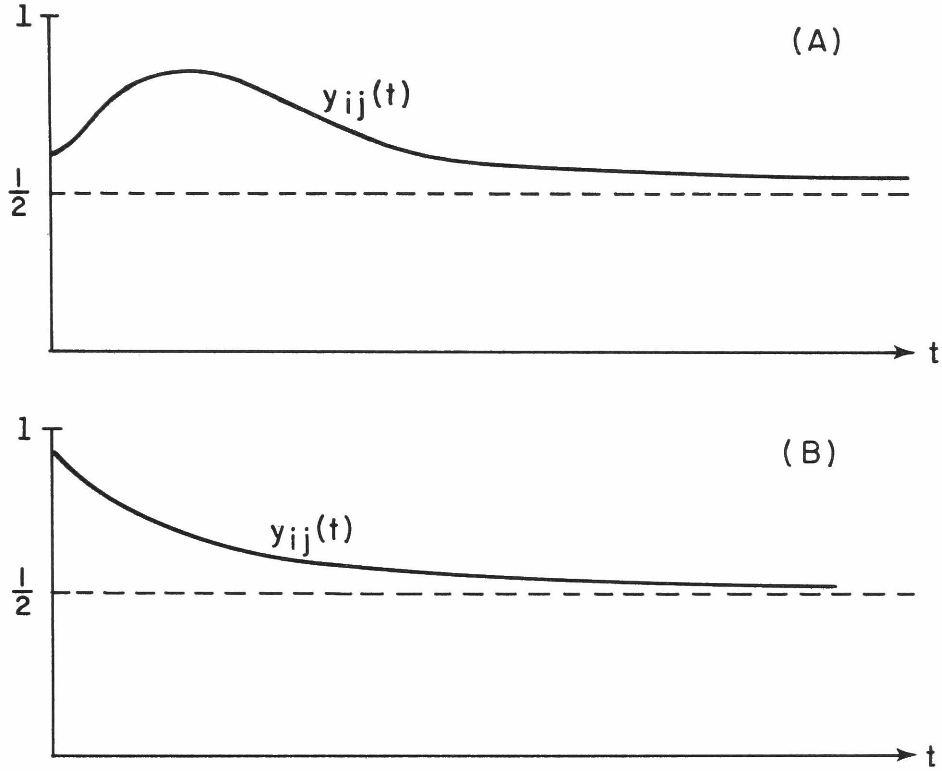


Fig. 17

(B) is guaranteed if, moreover, $x_k(0)z_{ij}(0) > x_j(0)z_{ik}(0)$. Thus after at most one start in the wrong direction due to an unfortunate choice of initial data, y_{ki} settles monotonically to its limit $\frac{1}{2}$ but does not reach this limit in finite time since $f_{ij}(0)g_{kij}(0) > 0$ implies $f_{ij}(t)g_{kij}(t) > 0$. This strong control on (*)'s oscillations is important to our prediction theory, since (*) can be called upon at <u>any</u> time to reproduce the ordering induced by prior inputs.

(b) The hypothesis $\alpha > \beta$ is not essential to the proof. Because $x = \sum_{k=1}^{3} x_k$ obeys the equation $\dot{x} = (\beta - \alpha)x$, (1) implies that $\lim_{t\to\infty} x_i(t) = 0$ if $\alpha > \beta$, that $\lim_{t\to\infty} x_i(t) = \infty$ if $\alpha < \beta$, and that $\sum_{k=1}^{3} x_k(t) = \text{constant}$ if $\alpha = \beta$. In all these cases, the ratios

$$X_i(t) = \frac{x_i(t)}{x_1(t) + x_2(t) + x_3(t)}$$

approach $\frac{1}{3}$ as $t \to \infty$ if $\sigma > 0$ and $\beta > 0$.

(c) The hypothesis $z_{ij}(0) = z_{ji}(0)$ is obviously equivalent to the hypothesis $z_{ij}(t) = z_{ji}(t)$, $t \geq 0$, even when (*) is not input-free. That is, the "reversibility" of the weights z_{ij} is a "geometrical" property of the graph.

(d) The condition $\sigma > 0$ is not superfluous in proving, as stated in (3), that $Q_i = \frac{1}{3}$ and $P_{jk} = \frac{1}{2}(1 - \delta_{jk})$. Indeed, when $\sigma < 0$ we shall prove as a corollary to Theorem 3.1 that

$$|P_{jk} - y_{jk}(0)| \leq 2 \log\left(1 + \frac{1}{\gamma_j} \int_0^\infty e^{-|\sigma| v} X_j(1 - X_j)\, dv\right),$$

where $\gamma_j = \dfrac{z^{(j)}(0)}{\beta x^2(0)}$. . Since $X_j(1 - X_j) \leq 1$, this implies

$$|P_{jk} - y_{jk}(0)| \leq 2 \log\left(1 + \frac{1}{\gamma_j |\sigma|}\right),$$

and thus the deviation of P_{ij} from $y_{ij}(0)$ can be made as small as we please by taking $\sigma < 0$ and $|\sigma|$ sufficiently large. In particular, if

$$\left|y_{jk}(0) - \tfrac{1}{2}\right| > 2 \log\left(1 + \frac{1}{\gamma_j |\sigma|}\right),$$

then $P_{jk} \neq \frac{1}{2}$.

PROOF: Theorem 3.1 is proved in three major steps. In step (I) below, we prove all the assertions of (2) and the existence of all limits $P_{jk} = \lim_{t\to\infty} y_{jk}(t)$ for arbitrary positive initial data satisfying $z_{ij} = z_{ji}$ by a direct method that makes use of special properties of (*). In (II) we use the existence of all P_{jk} to prove the existence of all limits $Q_i = \lim_{t\to\infty} X_i(t)$ by treating (3.1) as a system of linear equations in the unknown variables X_i with the almost constant coefficients y_{jk}. And in (III) we use the existence of all P_{jk} and Q_i to compute the possible values of these limits using various algebraic properties of (3.1)-(3.3). Then we apply some special facts found in (I) to show that these values are unique if $\sigma > 0$ and are as stated in the Theorem.

(I) To prove that all limits P_{ij} exist, we exhibit new unknown variables which obey equations that have a surprisingly linear form. Instead of treating directly the vector function $X(t) = (x_1(t), x_2(t), x_3(t))$ and the matrix function $Z(t) = \| z_{ij}(t) \|$ of unknowns, we shall use Lemma 1 to find two matrix functions $\| X_{ij}(t) \|$ and $\| Y_{ij}(t) \|$ of new unknowns such that each triple (X_{ij}, Y_{ij}, y_{ij}) will have the properties of the triple (X, Y, y) in the following basic lemma.

LEMMA 3.2. Let the real-valued functions $X(t)$, $Y(t)$, and $y(t)$ satisfy the following system of differential equations.

$$\dot{X} = aX + bY \tag{3.5}$$

$$\dot{Y} = cX + dY \tag{3.6}$$

$$(Y - X)^{\cdot} = e(Y - X) + fY \tag{3.7}$$

$$\dot{y} = g(Y - X) , \tag{3.8}$$

where the functions a, b, c, d, e, f, and g are continuous, and in addition the functions b, c, f, g are positive. Then the functions X, Y, $Y - X$, and $\dot{y}$ change sign at most once. X and Y do not change sign at all if $X(0)Y(0) \geq 0$, while $Y - X$ and $\dot{y}$ do not change sign at all if $X(0)Y(0) \geq 0$ and $(Y - X)(0)Y(0) \geq 0$. Moreover, $X(0)Y(0) > 0$ implies $X(t)Y(t) > 0$ for all

$t \geq 0$, while $X(0)Y(0) > 0$ and $(Y - X)(0)Y(0) > 0$ imply $(Y - X)(t)\, Y(t) > 0$ for all $t \geq 0$. If in addition y is bounded, then $\lim_{t\to\infty} y(t)$ exists.

The conclusions of the lemma concerning the sign changes of the functions X, Y, and $Y - X$ can be conveniently pictured in the (X, Y) plane as in Figure 18, where the arrows indicate the direction of the (X, Y) point through time.

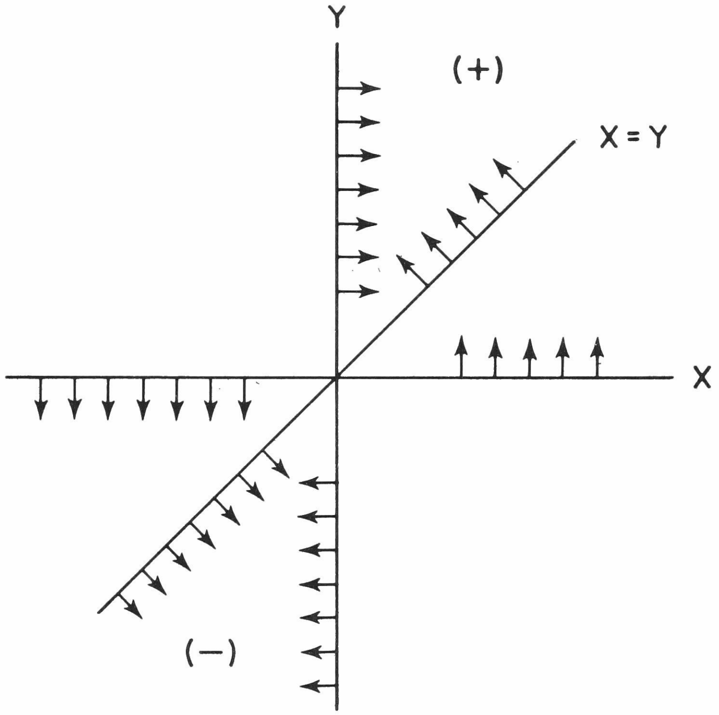


Fig. 18

If the (X, Y) point starts in the region $(+)$, then $y(t)$ is always monotone increasing, while if it starts in the region $(-)$, then $y(t)$ is always monotone decreasing.

PROOF: The vector field is non-trivial on the indicated critical lines. Also by uniqueness, the trajectory cannot arrive at 0 from another point. The conclusion is therefore clear for X, Y, and $X - Y$. It can also

easily be seen by integrating (3.5) - (3.7) using an exponential change of variable and then invoking nonnegativity of solutions; positivity is inessential.

Consider $\dot{y}$ in (3.8). Since g is positive, (B) implies that $\dot{y}$ changes sign at most once and not at all if $X(0)Y(0) \geq 0$ and $(Y-X)(0)Y(0) \geq 0$. We can therefore find a T such that $t \geq T$ implies $y(t)$ is a monotonic function. Thus if $y(t)$ is also bounded, then $\lim_{t \to \infty} y(t)$ exists. This completes the proof of Lemma 3.2.

Lemma 3.2 will prove the existence of all limits P_{jk} if we can find matrix functions $\| X_{jk}(t) \|$ and $\| Y_{jk}(t) \|$ with the properties of (3.5)-(3.8). We define these matrix functions in terms of the probability distributions x_{jk} and y_{jk} by letting $X_{jk} = \frac{1}{2} - x_{jk}$ and $Y_{jk} = \frac{1}{2} - y_{jk}$. It is easily seen that if these matrix functions satisfy the equations of Lemmas 3.2, then all the assertions of (2) in the statement of Theorem 3.1 are proved. We now proceed to show that these matrix functions do indeed have the desired properties. In the following discussion, only the triple (X_{21}, Y_{21}, y_{21}) will be considered. Our conclusions will carry over immediately to all triples (X_{ij}, Y_{ij}, y_{ij}) with $i \neq j$ by simply permuting indices.

(Ia) We seek an equation like (3.5) for X_{21} and Y_{21}. This equation is

$$\dot{X}_{21} = -A_{21}X_{21} + B_{21}Y_{21}$$

where

$$A_{21} = -\frac{\dot{X}_2}{1-X_2} + \beta\left(1 + \frac{y_{13}+y_{31}}{2}\right)$$

and

$$B_{21} = \beta\left[\frac{X_2}{1-X_2} + \frac{y_{13}}{2}\left(\frac{z^{(2)}}{z^{(3)}}\right)\right] > 0.$$

Letting $i = 3$ and then 1 in (3.1) and subtracting the two equations gives

$$(X_3 - X_1)^{\circ} = \beta\left[-(X_3 - X_1) + X_2(y_{23} - y_{21}) + X_1 y_{13} - X_3 y_{31}\right]$$

$$= \beta\left[-(X_3 - X_1) + X_2(y_{23} - y_{21}) + (X_1 - X_3) y_{13} + X_3(y_{13} - y_{31})\right],$$

or

$$(X_3 - X_1)^{\circ} = \beta\left[-(1 + y_{13})(X_3 - X_1) + X_2(y_{23} - y_{21}) + X_3(y_{13} - y_{31})\right]. \tag{3.9}$$

The left hand side of (3.9) is an antisymmetric function of the indices 3 and 1. We now seek an expression for the right hand side of (3.9) which is also antisymmetric in these indices. Permuting the indices 3 and 1 in (3.9) gives

$$(X_1 - X_3)^{\circ} = \beta\left[-(1 + y_{31})(X_1 - X_3) + X_2(y_{21} - y_{23}) + X_1(y_{31} - y_{13})\right]. \tag{3.10}$$

Subtract (3.10) from (3.9) and divide the resulting equation by 2. Then

$$(X_3 - X_1)^{\circ} = -H_{31}(X_3 - X_1) + \beta X_2(y_{23} - y_{21}) + \frac{\beta(1 - X_2)}{2}(y_{13} - y_{31}), \tag{3.11}$$

where

$$H_{31} = \beta\left(1 + \frac{y_{13} + y_{31}}{2}\right) = H_{13}.$$

The right hand side of (3.11) is clearly antisymmetric in the indices 1 and 3.

Using (3.11) we seek an equation for the derivative of X_{21} in terms of

X_{21} and Y_{21}. Since

$$\begin{aligned}
\mathbb{X}_{21} &= \frac{1}{2} - x_{21} \\
&= \frac{1}{2} - \frac{x_1}{x_1 + x_3} \\
&= \frac{x_3 - x_1}{2(x_1 + x_3)} \\
&= \frac{x_3 - x_1}{2(x - x_2)} \\
&= \frac{\mathbb{X}_3 - \mathbb{X}_1}{2(1 - \mathbb{X}_2)},
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbb{X}}_{21} &= \left[\frac{\mathbb{X}_3 - \mathbb{X}_1}{2(1 - \mathbb{X}_2)}\right]^{\bullet} \\
&= (\mathbb{X}_3 - \mathbb{X}_1)\frac{\dot{\mathbb{X}}_2}{2(1 - \mathbb{X}_2)^2} + \frac{1}{2(1 - \mathbb{X}_2)}(\mathbb{X}_3 - \mathbb{X}_1)^{\bullet} \\
&= \frac{\dot{\mathbb{X}}_2}{1 - \mathbb{X}_2}\mathbb{X}_{21} + \frac{1}{2(1 - \mathbb{X}_2)}\Big[-H_{31}(\mathbb{X}_3 - \mathbb{X}_1) + \beta\mathbb{X}_2(y_{23} - y_{21}) \\
&\qquad + \frac{\beta(1 - \mathbb{X}_2)}{2}(y_{13} - y_{31})\Big],
\end{aligned}$$

or

$$\dot{\mathbb{X}}_{21} = -\left(H_{31} - \frac{\dot{\mathbb{X}}_2}{1 - \mathbb{X}_2}\right)\mathbb{X}_{21} + \frac{\beta\mathbb{X}_2}{2(1 - \mathbb{X}_2)}(y_{23} - y_{21}) + \frac{\beta}{4}(y_{13} - y_{31}).$$

Consider the term $\frac{\beta\mathbb{X}_2}{2(1 - \mathbb{X}_2)}(y_{23} - y_{21})$ in this equation. Since $y_{23} = 1 - y_{21}$ and $Y_{21} = \frac{1}{2} - y_{21}$,

$$\frac{\beta\mathbb{X}_2}{2(1 - \mathbb{X}_2)}(y_{23} - y_{21}) = \frac{\beta\mathbb{X}_2}{1 - \mathbb{X}_2}\mathbb{Y}_{21}.$$

Letting $A_{21} = H_{31} - \frac{\dot{X}_2}{1 - X_2}$, we therefore find

$$\dot{X}_{21} = -A_{21} X_{21} + \frac{\beta X_2}{1 - X_2} Y_{21} + \frac{\beta}{4}(y_{13} - y_{31}). \tag{3.12}$$

Consider the term $\frac{\beta}{4}(y_{13} - y_{31})$ of (3.12) in the light of the hypothesis $z_{ij} = z_{ji}$.

$$\begin{aligned} y_{13} - y_{31} &= \frac{z_{13}}{z_{12} + z_{13}} - \frac{z_{31}}{z_{31} + z_{32}} \\ &= \frac{z_{13}}{z_{21} + z_{13}} - \frac{z_{13}}{z_{13} + z_{23}} \\ &= \frac{z_{13}(z_{23} - z_{21})}{(z_{21} + z_{13})(z_{13} + z_{23})} \\ &= \frac{z_{13}\, z^{(2)}}{z^{(1)} z^{(3)}} (y_{23} - y_{21}), \end{aligned}$$

and

$$\frac{\beta}{4}(y_{13} - y_{31}) = \frac{\beta}{2} y_{13} \frac{z^{(2)}}{z^{(3)}} Y_{21}. \tag{3.13}$$

Substituting (3.13) into (3.12) gives

$$\dot{X}_{21} = -A_{21} X_{21} + B_{21} Y_{21} \tag{3.14}$$

where

$$B_{21} = \beta \left[\frac{X_2}{1 - X_2} + \frac{y_{13}}{2}\left(\frac{z^{(2)}}{z^{(3)}}\right)\right].$$

(Ib) We now seek an equation like (3.6) for Y_{21} and X_{21}. This equation is

$$\dot{Y}_{21} = -G_2 Y_{21} + G_2 X_{21},$$

where

$$G_2 = \frac{\beta x_1 x^{(1)}}{z^{(1)}} > 0.$$

It is derived as follows. By (3.2) and (3.3),

$$\dot{y}_{21} = G_2\left(\frac{X_1}{1-X_2} - y_{21}\right)$$

where

$$G_2 = \frac{\beta x_1 x^{(1)}}{z^{(1)}} .$$

Since $Y_{21} = \frac{1}{2} - y_{21}$,

$$\dot{Y}_{21} = G_2\left(-\frac{X_1}{1-X_2} + \frac{1}{2} - Y_{21}\right)$$

Since

$$X_{21} = \frac{1}{2} - \frac{x_1}{x_1 + x_3}$$

$$= \frac{1}{2} - \frac{x_1}{x - x_2}$$

$$= \frac{1}{2} - \frac{X_1}{1 - X_2},$$

$$\dot{Y}_{21} = G_2\left(X_{21} - Y_{21}\right) . \tag{3.15}$$

(Ic) We now seek an equation like (3.7) for $X_{21} - Y_{21}$ and Y_{21}. This equation is

$$(Y_{21} - X_{21})^{\bullet} = -E_{21}(Y_{21} + X_{21}) + F_{21}Y_{21}$$

where

$$E_{21} = G_2 + A_{21} + \beta(y_{31} - y_{13})\, Y_{21}$$

and

$$F_{21} = \beta\left[y_{31}(y_{13} + y_{23}) + y_{21}y_{13}\right].$$

The strategy for deriving this equation is simple. We compute equations for $\dot{X}_{21}$ and $\dot{Y}_{21}$ by factoring out as many expressions $Y_{21} - X_{21}$ as possible. Then we combine these equations by subtraction. Since the equation for $\dot{Y}_{21}$, namely (3.15) is already in the desired form, we need only compute an equation for $\dot{X}_{21}$. We already have equation (3.14) for $\dot{X}_{21}$, and the new equation is obtained by merely rearranging terms in (3.14). By (3.14),

$$\dot{X}_{21} = - A_{21}X_{21} + B_{21}Y_{21} .$$

Using the normalization conditions $X_1 + X_2 + X_3 = 1$ and $y_{ij} + y_{ik} = 1$, $\{i, j, k\} = \{1, 2, 3\}$, we shall be able to rewrite (3.14) as

$$\dot{X}_{21} = - D_{21}(X_{21} - Y_{21}) - F_{21}Y_{21} \tag{3.16}$$

where F_{21} is <u>positive</u>. Subtracting (3.16) from (3.15) and setting $E_{21} = G_2 + D_{21}$ then gives

$$(Y_{21} - X_{21})^{\cdot} = -E_{21}(Y_{21} - X_{21}) + F_{21}Y_{21} \tag{3.17}$$

It therefore remains only to show that (3.16) can be found.

Letting $H_{21} = B_{21} - A_{21}$, (3.14) can be written as

$$\dot{X}_{21} = -A_{21}(X_{21} - Y_{21}) + H_{21}Y_{21} . \tag{3.18}$$

Consider H_{21}. Since

$$B_{21} = \beta\left[\frac{X_2}{1 - X_2} + \frac{y_{13}}{2}\left(\frac{z^{(2)}}{z^{(3)}}\right)\right]$$

and

$$A_{21} = \frac{-\dot{X}_2}{1-X_2} + \beta\left(1 + \frac{y_{13}+y_{31}}{2}\right),$$

where $\dot{x}_2 = \beta(-X_2 + X_1 y_{12} + X_3 y_{32})$,

it follows readily that

$$H_{21} = \beta\left(J_{21} + \frac{y_{13}}{2}\left(\frac{z^{(2)}}{z^{(3)}}\right)\right), \tag{3.19}$$

where

$$J_{21} = \frac{X_1 y_{12} + X_3 y_{32}}{1-X_2} - \left(1 + \frac{y_{13}+y_{31}}{2}\right).$$

We apply the normalization conditions to J_{21}.

$$J_{21} = \frac{X_1(1-y_{13}) + X_3(1-y_{31})}{1-X_2} - \left(1 + \frac{y_{13}+y_{31}}{2}\right)$$

$$= \frac{X_1+X_3}{1-X_2} - 1 - y_{13}\left(\frac{1}{2} + \frac{X_1}{1-X_2}\right) - y_{31}\left(\frac{1}{2} + \frac{X_3}{1-X_2}\right)$$

$$= -y_{13}\left(\frac{1}{2} + \frac{X_1}{1-X_2}\right) - y_{31}\left(\frac{1}{2} + \frac{X_3}{1-X_2}\right)$$

Since $X_{21} = \frac{1}{2} - \frac{X_1}{1-X_2}$ and $Y_{21} = \frac{1}{2} - y_{21}$, we find

$$\frac{1}{2} + \frac{X_1}{1-X_2} = \frac{1}{2} + y_{21} + \left(\frac{X_1}{1-X_2} - y_{21}\right)$$

$$= \frac{1}{2} + y_{21} + (Y_{21} - X_{21}),$$

and

$$\frac{1}{2} + \frac{X_3}{1-X_2} = \frac{1}{2} + \frac{1-X_2-X_1}{1-X_2}$$

$$= \frac{3}{2} - \frac{X_1}{1-X_2}$$

$$= \frac{3}{2} - y_{21} + (X_{21} - Y_{21}).$$

Thus

$$J_{21} = (y_{13} - y_{31})(X_{21} - Y_{21}) - y_{13}\left(\frac{1}{2} + y_{21}\right) - y_{31}\left(\frac{3}{2} - y_{21}\right). \quad (3.20)$$

Substituting (3.19) and (3.20) into (3.18) and rearranging terms gives

$$\dot{X}_{21} = -\tilde{D}_{21}(X_{21} - Y_{21}) - \tilde{F}_{21}Y_{21}, \quad (3.21)$$

where

$$\tilde{D}_{21} = A_{21} + \beta(y_{31} - y_{13})Y_{21}$$

and

$$\tilde{F}_{21} = \beta\left[y_{13}\left(\frac{1}{2} + y_{21}\right) + y_{31}\left(\frac{3}{2} - y_{21}\right) - \frac{y_{13}}{2}\left(\frac{z^{(2)}}{z^{(3)}}\right)\right].$$

We now show that $\tilde{F}_{21}$ is positive and thus that the identifications $\tilde{D}_{21} = D_{21}$ and $\tilde{F}_{21} = F_{21}$ can be made to give (3.16). For this computation, both the normalization conditions and the constraints $z_{ij} = z_{ji}$ are needed.

The terms in $\frac{1}{\beta}\tilde{F}_{21}$ can be rearranged to read

$$\frac{1}{\beta}\tilde{F}_{21} = K_{21} + y_{21}y_{13} + y_{31}\left(\frac{3}{2} - y_{21}\right),$$

where

$$
\begin{aligned}
K_{21} &= \frac{y_{13}}{2}\left(1-\frac{z^{(2)}}{z^{(3)}}\right)\\
&= \frac{y_{13}}{2}\left(\frac{z_{31}-z_{21}}{z^{(3)}}\right)\\
&= \frac{y_{13}}{2}\left(y_{31}-\frac{z_{21}}{z^{(3)}}\right)\\
&= \frac{1}{2}\left(y_{13}\,y_{31}-\frac{z_{13}\,z_{21}}{z^{(1)}\,z^{(3)}}\right)\\
&= \frac{1}{2}\left(y_{13}\,y_{31}-\frac{z_{12}\,z_{31}}{z^{(1)}\,z^{(3)}}\right)\\
&= \frac{1}{2}\,y_{31}\left(y_{13}-y_{12}\right)\\
&= y_{31}\left(\frac{1}{2}-y_{12}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{1}{\beta}\tilde{F}_{21} &= y_{31}\left(\frac{1}{2}-y_{12}\right)+y_{21}\,y_{13}+y_{31}\left(\frac{3}{2}-y_{21}\right)\\
&= y_{31}\left(2-y_{12}-y_{21}\right)+y_{21}\,y_{13}\\
&= y_{31}\left(y_{13}+y_{23}\right)+y_{21}\,y_{13}.
\end{aligned}
$$

$\tilde{F}_{21}$ is manifestly positive when it is written in this way, and we have therefore derived an equation of the form given in (3.16). This completes the derivation of an equation such as (3.17), for which

$$E_{21} = G_2 + D_{21} = G_2 + \tilde{D}_{21} = G_2 + A_{21} + \beta(y_{31} - y_{13})Y_{21}$$

and

$$F_{21} = \tilde{F}_{21} = \beta\left[y_{31}(y_{13} + y_{23}) + y_{21}y_{13}\right].$$

(Id) It remains only to produce an equation like (3.8) for $\dot{y}_{21}$ in terms of $Y_{21} - X_{21}$. This equation is (3.15). Since $y_{21} = \frac{1}{2} - Y_{21}$,

$$\dot{y}_{21} = G_2(Y_{21} - X_{21}), \tag{3.22}$$

where $G_2 > 0$. Equations (3.14), (3.15), (3.17), and (3.22) correspond to equations (3.5) - (3.8). By merely permuting indices, we can in the same way derive equations for all indices $\{i, j, k\} = \{1, 2, 3\}$. Thus all the assertions of (2) in Theorem 3.1 are immediate. Moreover, since y_{jk} is a bounded function, $P_{jk} = \lim_{t\to\infty} y_{jk}(t)$ exists for all $j \neq k$.

(II) We now use the existence of all P_{jk} to prove the existence of all $Q_i = \lim_{t\to\infty} X_i(t)$ by treating (3.1) as a linear system of equations in the unknown variables X_i with the almost constant coefficients y_{jk}. The first step in this treatment is to reduce (3.1) from a 3-by-3 system of equations to a 2-by-2 system by utilizing the normalization condition $X_1 + X_2 + X_3 = 1$. After doing this, the proof becomes straightforward.

By (3.1),

$$\begin{aligned}\dot{X}_1 &= \beta(-X_1 + X_2 y_{21} + (1 - X_1 - X_2) y_{31}) \\ &= -\beta(1 + y_{31})X_1 + (y_{21} - y_{31})X_2 + y_{31}.\end{aligned}$$

Letting $m_{ij} = y_{ij} - P_{ij}$, this becomes

$$\dot{X}_1 = -\beta(1 + P_{31} + m_{31})X_1 + \beta(P_{21} - P_{31} + m_{21} - m_{31})X_2 + \beta y_{31}.$$

Similarly,

$$\dot{X}_2 = -\beta(1 + P_{32} + m_{32})X_2 + \beta(P_{12} - P_{32} + m_{12} - m_{32})X_1 + \beta y_{32}.$$

Letting $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, these equations can be written in matrix form as

$$\dot{X} = (A + B(t))X + C(t), \tag{3.23}$$

where

$$A = \beta \begin{pmatrix} -(1+P_{31}) & P_{21}-P_{31} \\ P_{12}-P_{32} & -(1+P_{32}) \end{pmatrix}$$

$$B(t) = \beta \begin{pmatrix} -m_{31}(t) & m_{21}(t)-m_{31}(t) \\ m_{12}(t)-m_{32}(t) & -m_{32}(t) \end{pmatrix},$$

and

$$C(t) = \beta \begin{pmatrix} y_{31}(t) \\ y_{32}(t) \end{pmatrix}.$$

Since $\lim_{t\to\infty} y_{ij}(t) = P_{ij}$, $\lim_{t\to\infty} B(t) = 0$ and $\lim_{t\to\infty} C(t) = \beta \begin{pmatrix} P_{31} \\ P_{32} \end{pmatrix}$.

Since $X(t)$ is bounded, $\lim_{t\to\infty} B(t)X(t) = 0$ and (3. 23) can be written in the form

$$\mathring{X} = AX + f(t) \tag{3. 24}$$

where

$$f(t) = B(t)X(t) + C(t)$$

and

$$\lim_{t\to\infty} f(t) = \beta \begin{pmatrix} P_{31} \\ P_{32} \end{pmatrix}.$$

We now show that the eigenvalues of the matrix A have negative real parts. From this and the existence of the limit $\lim_{t\to\infty} f(t)$, it follows from (3. 24) by elementary arguments ([8]) that the limits $Q_i = \lim_{t\to\infty} X_i(t)$

exist for all $i = 1, 2, 3$. To show that the eigenvalues λ_1 and λ_2 of A have negative real parts, we need only the elementary formulas $\lambda_1\lambda_2 = \det A$ and $\lambda_1 + \lambda_2 = \operatorname{tr} A$. $\det A$ is positive since

$$\begin{aligned}
\det A &= \beta^2[(1+P_{31})(1+P_{32}) - (P_{21}-P_{31})(P_{12}-P_{32})] \\
&= \beta^2[1 + P_{31} + P_{32} + P_{21}(P_{32} - P_{12}) + P_{31}P_{12}] \\
&= \beta^2[2 + P_{21}(P_{32} - P_{12}) + P_{31}P_{12}] \\
&\geq \beta^2 > 0 .
\end{aligned}$$

$\operatorname{Tr} A$ is negative since

$$\operatorname{tr} A = -\beta(1 + P_{31} + 1 + P_{32}) = -3\beta < 0.$$

The eigenvalues λ_1 and λ_2 are either conjugate complex numbers or they are both real. In the former case, $\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = \frac{1}{2} \operatorname{tr} A < 0$. In the latter case, λ_1 and λ_2 have the same sign since $\lambda_1\lambda_2 > 0$. This sign is negative since $\lambda_1 + \lambda_2 < 0$. This completes the proof of the existence of the limits Q_i.

(III) Having established the existence of all limits Q_i and P_{jk}, we now show that $Q_i = \frac{1}{3}$ and $P_{jk} = \frac{1}{2}(1 - \delta_{jk})$ if $\sigma = u + 2(\beta - \alpha) > 0$. The first step in this proof is to show that the following relations hold.

$$Q_i \leq \frac{1}{2} , \tag{3. 25}$$

$$Q_i = Q_j P_{ji} + Q_k P_{ki} , \tag{3. 26}$$

and $$P_{jk} = \frac{Q_k}{1 - Q_j} , \quad j \neq k, \quad \text{if } Q_j > 0 . \tag{3. 27}$$

(3. 25) and (3. 26) shall be seen not to depend on the hypothesis $\sigma > 0$.

The inequality (3.25) follows from two facts: $X_i(0) < 1$ because all initial data is positive, and $\dot{X}_i < 0$ whenever $X_i > \frac{1}{2}$ because

$$\begin{aligned}\dot{X}_i &= \beta(-X_i + X_j y_{ji} + X_k y_{ki})\\ &\leq \beta(-X_i + X_j + X_k)\\ &= 2\beta(\tfrac{1}{2} - X_i)\ .\end{aligned}$$

The equations (3.26) follow directly by letting $t \to \infty$ in (3.1) and using the existence of all limits Q_i and P_{jk} to conclude that $\lim_{t\to\infty} \dot{X}_i(t)$ exists and equals $\beta(-Q_i + Q_j P_{ji} + Q_k P_{ki})$. Since X_i is bounded, $\lim_{t\to\infty} \dot{X}_i(t) = 0$. (3.2) is now immediate since $\beta \neq 0$.

Equation (3.27) can be derived as follows. By (3.25) and the hypothesis $Q_i > 0$, $0 < Q_i(1-Q_i)$. Thus by (3.3)

$$\begin{aligned}\lim_{t\to\infty} G_j(t) &= \frac{\lim_{t\to\infty} x_j(1-x_j)}{\lim_{t\to\infty}\left(\gamma_j e^{-\sigma t} + \int_0^t e^{-\sigma(t-s)} x_j(1-x_j)\,ds\right)}\\ &= \frac{Q_j(1-Q_j)}{Q_j(1-Q_j)/\sigma}\\ &= \sigma > 0\end{aligned}$$

Letting $t \to \infty$ in (3.2) therefore shows that $\lim_{t\to\infty} \dot{y}_{jk}(t)$ exists and equals

$$\sigma\left(\frac{Q_k}{1-Q_j} - P_{jk}\right).$$

Since y_{jk} is bounded, $\lim_{t\to\infty} \dot{y}_{jk}(t) = 0$. Since $\sigma > 0$, (3.27) follows immediately.

Using (3.25), (3.26), and (3.27), we now show that the possible values of Q_i and P_{jk} can be grouped into two cases if $\sigma > 0$.

Case 1. If all $Q_i > 0$, then $Q_i = \frac{1}{3}$ and $P_{jk} = \frac{1}{2}(1-\delta_{jk})$. This is proved by substituting (3.27) into (3.26). Then

$$Q_i = Q_i \left(\frac{Q_j}{1-Q_j} + \frac{Q_k}{1-Q_k} \right).$$

Since $Q_i > 0$,

$$1 = \frac{Q_j}{1-Q_j} + \frac{Q_k}{1-Q_k}.$$

This is true for all $j \neq k$. Thus

$$\frac{Q_1}{1-Q_1} = \frac{Q_2}{1-Q_2} = \frac{Q_3}{1-Q_3},$$

$Q_1 = Q_2 = Q_3$, and since $\sum_{k=1}^{3} Q_k = 1$, $Q_i = \frac{1}{3}$. By (3.27)

$$P_{jk} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$$

if $j \neq k$.

Case 2. If one $Q_i = 0$, then $Q_j = Q_k = \frac{1}{2}$, $P_{jk} = P_{kj} = 1$, and $P_{ji} = P_{ki} = 0$. (No more than one $Q_i = 0$ since then some $Q_k = 1$ which contradicts (3.25).) By (3.26)

$$0 = Q_i = Q_j P_{ji} + Q_k P_{ki}.$$

Since all limits are nonnegative,

$$0 = Q_j P_{ji} = Q_k P_{ki}.$$

Since $Q_j > 0$ and $Q_k > 0$, $0 = P_{ji} = P_{ki}$ and $1 = P_{jk} = P_{kj}$. By (3.26)

$$Q_j = Q_i P_{ij} + Q_k P_{kj} = Q_k P_{kj} = Q_k.$$

Since $1 = \sum_{m=1}^{3} Q_m = Q_j + Q_k$, $Q_j = Q_k = \frac{1}{2}$.

To complete the proof that $Q_i = \frac{1}{3}$ and $P_{jk} = \frac{1}{2}(1-\delta_{jk})$ if $\sigma > 0$, we now show that Case 2 cannot arise by employing special facts from (I) concerning the manner in which the functions y_{ji} and y_{ki} approach their limits. Suppose Case 2 holds. By (I) y_{ji} and y_{ki} are monotonic functions for $t \geq T$. y_{ji} and y_{ki} are also nonnegative functions and their limits P_{ji} and P_{ki} are zero by hypothesis. Thus y_{ji} and y_{ki} are monotonically decreasing functions for $t \geq T$; that is $\dot{y}_{ji} \leq 0$ and $\dot{y}_{ki} \leq 0$ for $t \geq T$. By (3.2) this means $y_{ji} \geq \frac{X_i}{1-X_j}$ and $y_{ki} \geq \frac{X_i}{1-X_k}$ for $t \geq T$. We use these inequalities to estimate $\dot{X}_i$ for large t. In particular, we shall be able to find a $T_1 \geq T$ such that $\dot{X}_i \geq \frac{\beta}{2} X_i > 0$ for $t \geq T_1$. Since X_i is positive, X_i can therefore never achieve a zero limit. But $Q_i = 0$, by hypothesis. This contradiction shows that only Case 1 can arise.

To establish the desired estimate for X_i, consider (3.1) for $t \geq T$.

$$\begin{aligned}\dot{X}_i &= \beta(-X_i + X_j y_{ji} + X_k y_{ki}) \\ &= \beta\left[\left(\frac{X_j}{1-X_j} + \frac{X_k}{1-X_k} - 1\right) X_i + X_j\left(y_{ji} - \frac{X_i}{1-X_j}\right) + X_k\left(y_{ki} - \frac{X_i}{1-X_k}\right)\right] \\ &\geq \beta\left(\frac{X_j}{1-X_j} + \frac{X_k}{1-X_k} - 1\right) X_i .\end{aligned}$$

By the hypothesis $Q_j = Q_k = \frac{1}{2}$, it follows that

$$\lim_{t\to\infty} \frac{X_j}{1-X_j} = \lim_{t\to\infty} \frac{X_k}{1-X_k} = 1 .$$

Thus there exists a T_1 such that

$$\dot{X}_i \geq \frac{\beta}{2} X_i \quad \text{for } t \geq T_1 .$$

This completes the proof of Theorem 3.1.

In Remark (a) to Theorem 3.1, we saw that each function y_{ij} approaches the limit $\frac{1}{2}$ monotonically except possibly for one peak in its graph. We now discuss how the analogous functions $x_{ij} = \frac{x_j}{x_j + x_k}$, where $\{i, j, k\} = \{1, 2, 3\}$ approach the limit $\frac{1}{2}$ in a time interval $[T_2, \infty)$ after all the functions X_{ij}, Y_{ij}, and $Y_{ij} - X_{ij}$ have made their single sign change. We shall find that there are only two possible avenues of approach.

For example, suppose that $y_{ij}(T_2) > \frac{1}{2}$. Then one of two graphs in Figure 19 holds.

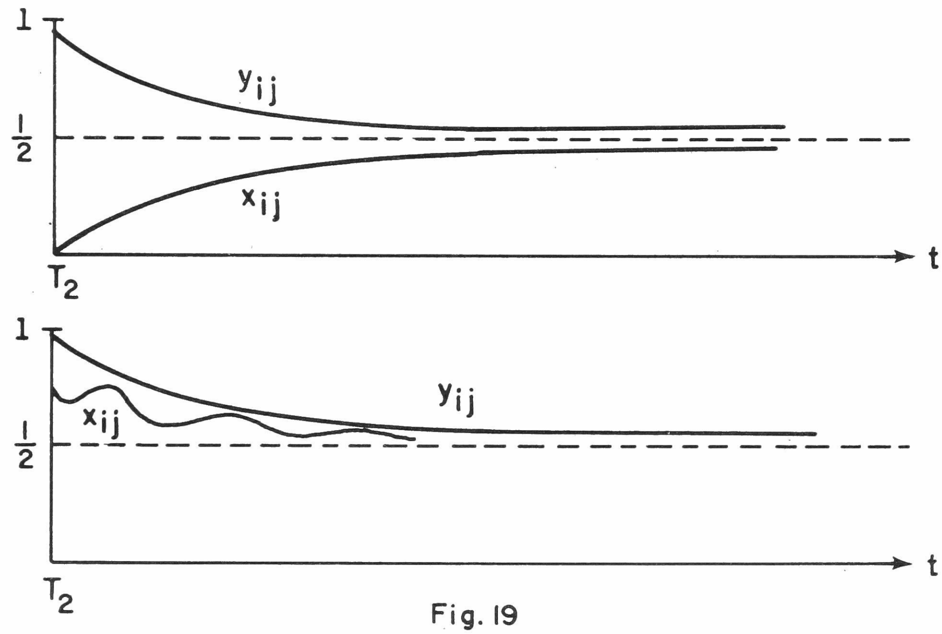


Fig. 19

That is, either $x_{ij}(T_2) < \frac{1}{2}$ and x_{ij} increases monotonically to $\frac{1}{2}$, or $x_{ij}(T_2) \geq \frac{1}{2}$ and the oscillations of x_{ij}, if any, are squeezed between y_{ij} and $\frac{1}{2}$ as $y_{ij} \rightarrow \frac{1}{2}$.

If $y_{ij}(T_2) < \frac{1}{2}$, the symmetric situation prevails and is diagrammed in Figure 20.

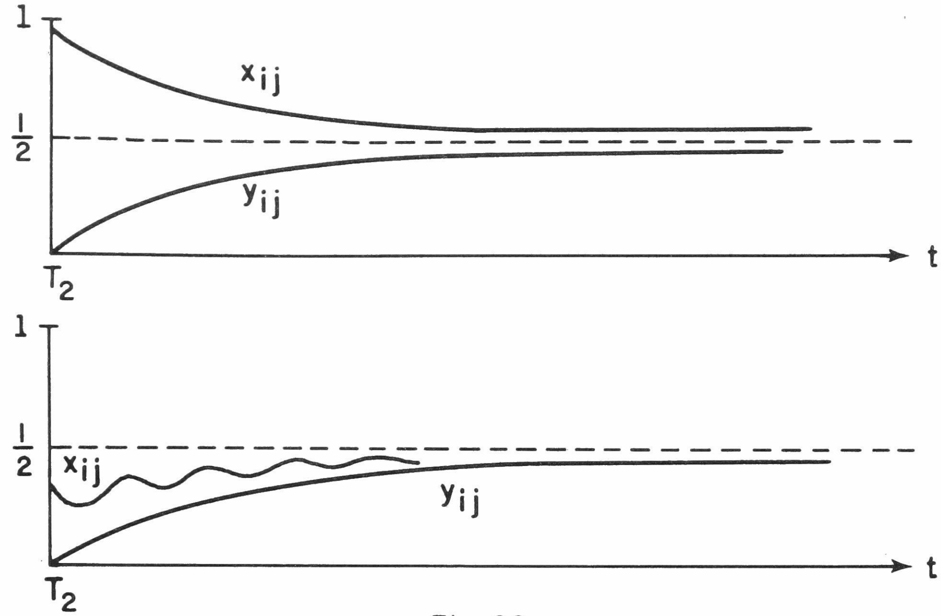


Fig. 20

We prove these alternatives in the following corollary.

COROLLARY 31. There exists a T_2 such that exactly one of the following alternatives holds for all $t \geq T_2$ if $\sigma > 0$:

1) $y_{ij} \geq x_{ij} \geq \frac{1}{2}$

2) $\frac{1}{2} \geq x_{ij} \geq y_{ij}$

3) $y_{ij} \geq \frac{1}{2} \geq x_{ij}$ and x_{ij} is monotone increasing.

4) $x_{ij} \geq \frac{1}{2} \geq y_{ij}$ and x_{ij} is monotone decreasing.

PROOF. By (I), X_{ij}, Y_{ij}, and $X_{ij} - Y_{ij}$ do not change sign for $t \geq T$ if T is chosen sufficiently large. For example, we can have $X_{ij} \geq 0$ for $t \geq T$. This is the same as $x_{ij} \geq \frac{1}{2}$ for $t \geq T$. Similarly $Y_{ij} \geq 0$ for $t \geq T$ gives $y_{ij} \geq \frac{1}{2}$ for $t \geq T$. And $X_{ij} - Y_{ij} \geq 0$ for $t \geq T$ gives $x_{ij} \geq y_{ij}$ for $t \geq T$. By examining all possible inequalities in this way, it follows that the relative magnitudes of x_{ij}, y_{ij}, and $\frac{1}{2}$ are fixed for $t \geq T$.

We also know that y_{ij} is a monotonic function for $t \geq T$ and that $P_{ij} = \frac{1}{2}$. Thus if (say) $y_{ij}(T) \geq \frac{1}{2}$, then y_{ij} decreases monotonically to $\frac{1}{2}$ for $t \geq T$. This means $\dot{y}_{ij} \leq 0$ for $t \geq T$. Since $\dot{y}_{ij} = G_i(x_{ij} - y_{ij})$, $y_{ij} \geq x_{ij}$ for $t \geq T$. This situation admits two possible subcases. Either $y_{ij} \geq x_{ij} \geq \frac{1}{2}$ for $t \geq T$, or $y_{ij} \geq \frac{1}{2} \geq x_{ij}$ for $t \geq T$. These are cases (1) and (3) in the statement of the corollary. Cases (2) and (4) arise if $y_{ij}(T) \leq \frac{1}{2}$. Obviously cases (1)-(4) exhause all the possibilities.

In cases (1) and (2), x_{ij} is bounded by $\frac{1}{2}$ and by y_{ij} for $t \geq T$ while y_{ij} converges monotonically to $\frac{1}{2}$. Thus x_{ij} is forced into an ever smaller interval as t increases and its oscillations, if any, become smaller and smaller. We shall now show that in cases (3) and (4), no oscillations whatsoever occur in x_{ij} if t is taken sufficiently large. Since $\dot{x}_{ij} = -\dot{X}_{ij}$, it suffices to show that for all large t, $\dot{X}_{ij}$ is either nonpositive or nonnegative.

Consider case (3) for specificity. By (3.14),

$$\dot{X}_{21} = -A_{21}X_{21} + B_{21}Y_{21}$$

where $B_{21} > 0$ and

$$A_{21} = \frac{-\dot{X}_2}{1 - X_2} + \beta\left(1 + \frac{y_{13} + y_{31}}{2}\right)$$

Suppose we can show that there is a $T_2 \geq T$ such that A_{21} is positive for $t \geq T_2$. Since we are in case (3), $X_{21} = \frac{1}{2} - x_{21} \geq 0$ and $Y_{21} = \frac{1}{2} - y_{21} \leq 0$. for $t \geq T_2$. Thus $\dot{x}_{21} = -\dot{X}_{21} \geq 0$ for $t \geq T_2$ and x_{21} increases monotonically to $\frac{1}{2}$. An identical argument shows that x_{21} decreases monotonically to $\frac{1}{2}$ in case (4). We now show that such a T_2 exists.

$$A_{21} = \beta\left[\frac{X_2 - X_1 y_{12} - X_3 y_{32}}{1 - X_2} + 1 + \frac{y_{13} + y_{31}}{2}\right]$$

$$= \beta\left[\frac{1 - X_1 y_{12} - X_3 y_{32}}{1 - X_2} + \frac{y_{13} + y_{31}}{2}\right]$$

$$\geq \beta\left[\frac{1 - \frac{1}{2}(X_1 + X_3) - X_1\left(y_{12} - \frac{1}{2}\right) - X_3\left(y_{32} - \frac{1}{2}\right)}{1 - X_2}\right]$$

$$= \beta\left[\frac{1}{1 - X_2} - \frac{1}{2} - \frac{X_1\left(y_{12} - \frac{1}{2}\right) - X_3\left(y_{32} - \frac{1}{2}\right)}{X_1 + X_3}\right]$$

$$\geq \beta\left(\frac{1}{2} - \left|y_{12} - \frac{1}{2}\right| - \left|y_{32} - \frac{1}{2}\right|\right).$$

Since $\lim_{t\to\infty} y_{12}(t) = \lim_{t\to\infty} y_{32}(t) = \frac{1}{2}$, we can obviously choose a T_2 such that

$$A_{21} \geq \frac{\beta}{4} > 0 \quad \text{for} \quad t \geq T_2,$$

and the proof is complete.

Theorem 3.1 shows that the limits of the ratios $X_i(t)$ and $y_{jk}(t)$ can be made unique by choosing $\sigma > 0$. We now show that this choice of σ is not a superfluous condition by proving the following corollary.

<u>COROLLARY 3.2.</u> For all choices of σ and arbitrary nonnegative initial data,

$$|P_{jk} - y_{jk}(0)| \le 2 \log\left(1 + \frac{1}{\gamma_j}\int_0^\infty e^{\sigma v} X_j(1-X_j)\,dv\right).$$

This corollary exerts a constraint on P_{jk} only if $\sigma < 0$, since then, as shown in Remark (D) above,

$$|P_{jk} - y_{jk}(0)| \le 2 \log\left(1 + \frac{1}{\gamma_j|\sigma|}\right).$$

We can thus make the deviation of P_{jk} from $y_{jk}(0)$ as small as we please by choosing $\sigma < 0$ and $|\sigma|$ sufficiently large.

<u>PROOF</u>. By (3.2) and (3.3),

$$\begin{aligned}|\dot{y}_{jk}| &= G_j\left|\frac{X_k}{X_i + X_k} - y_{jk}\right| \\ &\le G_j\left(\frac{X_k}{X_i + X_k} + y_{jk}\right) \\ &\le 2G_j,\end{aligned}$$

where

$$G_j(t) = \frac{d}{dt}\log\left(\gamma_j + \int_0^t e^{\sigma v} X_j(1-X_j)\,dv\right),$$

and $\gamma_j = \dfrac{z^{(j)}(0)}{\beta x^2(0)}$. Thus for every $t \ge 0$,

$$\begin{aligned}|y_{jk}(t) - y_{jk}(0)| &\le \int_0^t |\dot{y}_{jk}(w)|\,dw \\ &\le 2\int_0^t \frac{d}{dw}\log\left(\gamma_j + \int_0^w e^{\sigma v} X_j(1-X_j)\,dv\right) \\ &\le 2\log\left(1 + \frac{1}{\gamma_j}\int_0^t e^{\sigma v} X_j(1-X_j)\,dv\right).\end{aligned}$$

In particular, letting $t \to \infty$ gives

$$|P_{jk} - y_{jk}(0)| \leq 2 \log \left(1 + \frac{1}{\gamma_j} \int_0^\infty e^{\sigma v} X_j (1 - X_j) dv\right).$$

Theorem 3. 1 and Corollary 3. 2 have the following heuristic consequences. If $\sigma > 0$, then the complete 3-graph without loops "forgets" its initial data. If $\sigma < 0$ and $|\sigma|$ is sufficiently large, then the graph "remembers" its initial data to an arbitrary degree of accuracy. Even when $\sigma < 0$ and $|\sigma| >> 0$, the "equilibrium" equations

$$\frac{1}{2} \geq Q_i = Q_j P_{ji} + Q_k P_{ki}, \quad \{i, j, k\} = \{1, 2, 3\},$$

hold. This means that although the ratios $y_{jk}(t)$ move very little, the ratios $X_i(t)$ adjust themselves as much as is required to reach an "equilibrium" state.

Consider the case $\sigma < 0$. Then $u < 2(\alpha - \beta)$. Only the case $\alpha - \beta > 0$ is of prediction theoretic interest, since then the outputs $x_i(t)$ approach zero as $t \to \infty$. β gives the rate at which x_i is "excited" by an input from another vertex, and α gives the rate of x_i decay as the effects of this input wear off. u gives the rate at which the "cross-correlation" $\beta x_j(t-\tau) x_k(t)$ of pulses $\beta x_j(t-\tau)$ and $x_k(t)$ in edge e_{jk} wears off. Suppose that we have constructed a system in which both the excitation rate and the decay rate of the states x_i are rapid and of the same order of magnitude (i. e., $\alpha \cong \beta >> 0$). Such a system has the virtue that its response to external perturbations I_i is rapid and does not introduce large "inertial" effects. Then $\sigma < 0$ implies $u < 2(\alpha - \beta) \cong 0$. Thus the rate of decay of the cross-correlations must be very slow if the excitation and decay rates of the states are large and approximately symmetric. Otherwise, the system will forget everything that it is taught.

2. A RELATIONSHIP BETWEEN MEASUREMENT, LINEARITY, AND REVERSIBILITY.

A special case of Theorem 3.1 is considered below along with a heuristic interpretation. (*) is given at $t = 0$ with $\sigma > 0$ and uniform initial data (i.e., $z_{ij}(0) = \delta > 0$, $i \neq j$, and $x_i(0) = \gamma > 0$, $i = 1, 2, 3$). v_i is perturbed by inputs within the finite time interval $(T_1, T_2]$, where $0 < T_1 < T_2 < \infty$, and (*) is input-free in (T_2, ∞).

Clearly, $x_i(t) = \frac{1}{3}x(t)$ for $t \in [0, T_1]$, where $x = \sum_{k=1}^{3} x_k$. Thus $\dot{x}_i(t) = (\beta - \alpha)x_i(t)$ for $t \in [0, T_1]$, so that the output from each v_i is linear. Moreover, $z_{ij}(t)$ is independent of i and j, $i \neq j$, for $t \in [0, T_1]$, so $y_{ij}(t) = \frac{1}{2} = y_{ji}(t)$. That is, the flow from v_i to v_j and from v_j to v_i is globally reversible ("globally" because y_{ij} depends on all indices i, j, k).

In $(T_1, T_2]$, the output from (*) is obviously nonlinear. Moreover $y_{12}(t) \neq y_{21}(t)$, so that the flow between v_1 and v_2 is globally irreversible. By contrast, $z_{ij} = z_{ji}$ so that the flow is still locally reversible ("locally" because z_{ij} depends only on indices i and j).

In (T_2, ∞), (*) is input-free and $z_{ij}(T_2) = z_{ji}(T_2)$. By Theorem 3.1, $\lim_{t \to \infty} \frac{x_i(t)}{\frac{1}{3}x(t)} = 1$, where $\frac{1}{3}x$ obeys a linear equation. That is, the output of each v_i is eventually linear once again. Moreover, $\lim_{t \to \infty} \frac{y_{ij}(t)}{y_{ji}(t)} = \frac{1/2}{1/2} = 1$. That is, the flows within (*) are eventually globally reversible once again.

The input to v_1 for $t \in (T_1, T_2]$ is interpreted as a measurement performed by an experimenter studying (*). This example therefore illustrates that a measurement can transform a linear and globally reversible system into a nonlinear and globally irreversible system, but that linearity and global reversibility are gradually restored as the effect of the measurement wears off. The measurement does not affect local reversibility.

3. THE OUTPUT OF AN INPUT-FREE GRAPH IS NOT A GOOD INDEX OF ITS MEMORY.

In a complete 3-graph without loops, $\sigma > 0$ implies that the ratios $y_{jk}(t)$ approach the same limits $P_{jk} = \frac{1}{2}(1-\delta_{jk})$ no matter what positive and locally reversible initial values they originally possessed. On the other hand, when $\sigma < 0$ and $|\sigma| >> 0$, the ratios $y_{jk}(t)$ deviate very little from their initial values. These remarks have a prediction theoretic interpretation in the case $\alpha > \beta$, since then the outputs x_i approach zero as $t \to \infty$ if the graph is input-free for all large times. In this case, we can say that a graph with $\sigma > 0$ "forgets" its past as $t \to \infty$ in the sense that a test input pulse occurring at vertex v_i at a large time T will produce approximately equal outputs $x_j(t)$ and $x_k(t)$ from the vertices v_j and v_k for $t \geq T$ and all $\{i, j, k\} = \{1, 2, 3\}$. In a similar fashion, we can say that the graph at least partially "remembers" its past when $\sigma < 0$. In both these cases, the outputs converge exponentially to zero as $t \to \infty$ if the graph is not perturbed at large times.

An experimentalist passively measuring from a graph with $\sigma > 0$ and one with $\sigma < 0$ and $|\sigma| >> 0$ will be inclined to think that both graphs are forgetting his prior perturbations as the outputs which he measures approach zero. In the case of the graph with $\sigma > 0$, he is correct. In the case of the graph with $\sigma < 0$ and $|\sigma| >> 0$, he is wrong. In Chapter II, we also saw that an input-free outstar never forgets its initial data no matter how its positive coefficients α, β, and u are chosen. Nonetheless, the outputs of an input-free outstar always converge exponentially to zero. These three cases amply illustrate that the absolute size of the outputs from an input-free graph in no way indicates the way in which the graph will later respond to a test input pulse.

A qualitative conclusion can be drawn from the contrasts of the previous paragraph. The "dynamical mechanism" of an outstar and of a complete 3-graph without loops is the same. The two systems differ only

in the choice of their coefficient matrix P and possibly in the choice of their coefficients α, β, and u. That is, these systems differ only in the choice of their underlying "geometry". We therefore see that changing the geometry of a graph can qualitatively change the way in which it learns from experience. In the next section, we shall see that adding loops to an input-free complete graph enables it to "remember" its initial data even when $\sigma > 0$, in sharp contrast to the graph without loops.

4. AN INPUT-FREE GRAPH WITH LOOPS.

In this section, we consider the system with the coefficient matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and $\tau = 0$. This system obeys the equations

$$\begin{aligned} \dot{x}_i(t) &= -\alpha x_i(t) + \beta \sum_{m=1}^{2} x_m(t) y_{mi}(t), \\ y_{jk}(t) &= z_{jk}(t)/(z_{jj}(t) + z_{jk}(t)), \qquad (*) \\ \text{and} \quad \dot{z}_{jk}(t) &= -uz_{jk}(t) + x_j(t)x_k(t), \end{aligned}$$

for all $i, j, k = 1, 2$. Its coefficient graph is the complete 2-graph with loops of Figure 15.

This graph is closely related to the complete 3-graph without loops of Figure 14. In both graphs, every vertex is connected once to every other vertex. Moreover, in both graphs each vertex is touched by exactly two arrowheads. Thus an observer sitting on a vertex cannot tell from its immediate geometry which graph he is in. Nonetheless, the dynamics of the two graphs are dramatically different. Our main result for the 2-graph is

the following theorem, which discusses the ratios $y_{jk}(t)$ and $X_i(t) = \dfrac{x_i(t)}{x_1(t)+x_2(t)}$.

THEOREM 3.2. Let (*) be given with arbitrary nonnegative initial data and arbitrary positive coefficients α, β, and u. Then

(1) The limits $Q_i = \lim_{t\to\infty} X_i(t)$ and $P_{jk} = \lim_{t\to\infty} y_{jk}(t)$ exist and obey the equations

$$Q_i P_{ij} = Q_j P_{ji}, \quad \{i, j\} = \{1, 2\}.$$

(2) The functions $f_{jk} = X_k - y_{jk}$ and $\dot{y}_{jk}$ change sign at most once and not at all if $f_{jk}(0) f_{kk}(0) \leq 0$, $\{j, k\} = \{1, 2\}$.

(3) The initial data can be chosen so that the limits Q_i and P_{jk} satisfy the equations $P_{ji} = Q_i$, where the limits Q_1, Q_2 can form an arbitrary probability distribution.

If moreover $\sigma \equiv u + 2(\beta - \alpha) > 0$, then

(4) The limits Q_i and P_{jk} always satisfy the equations $P_{ji} = Q_i$.

Theorem 3.2 differs dramatically from Theorem 3.1 because the ratios Q_i and P_{jk} are not uniquely determined in Theorem 3.2 even if $\sigma > 0$. Adding the assumption $z_{12}(0) = z_{21}(0)$ in no way changes this situation, since the relative size of the initial values $z_{11}(0)$ and $z_{22}(0)$ is not affected by this condition. Theorems 3.1 and 3.2 together show that a vertex "knows" whether or not the flow it receives comes from another vertex or itself, in the sense that the limiting behavior of its vertex function differs in the two cases. When the complete graph has no loops and $\sigma > 0$, the limiting ratios "forget" their initial data. Adding loops enables the graph to "remember" its initial data.

PROOF. The strategy of the proof is essentially the same as that of Theorem 3.1. We therefore exhibit only the relevant formulas and presuppose familiarity with previous arguments to immediately draw conclu-

sions from these formulas. We also assume that all initial data are positive unless otherwise stated, since all other cases can be handled with ease once this case is understood.

(I) The first step in the proof is to derive equations for the ratios X_i and y_{jk}. These are readily seen to be the following by the usual manipulations

$$\dot{X}_i(t) = \beta(-X_i(t) + X_i(t)y_{ii}(t) + X_j(t)y_{ji}(t)), \ \{i, j\} = \{1, 2\} \tag{3.28}$$

and

$$\dot{y}_{ji}(t) = B_j(t)(X_i(t) - y_{ji}(t)), \quad j, i = 1, 2 \tag{3.29}$$

where

$$B_j = \frac{\beta x_j x}{z^{(j)}}, \quad x = x_1 + x_2, \quad \text{and} \quad z^{(j)} = z_{jj} + z_{ji}.$$

Since $X_1 + X_2 = 1$, (3.29) gives

$$\dot{X}_i = \beta[X_i(y_{ii} - X_i) + X_j(y_{ji} - X_i)]. \tag{3.30}$$

(II) Using these equations, we show that the limits P_{ij} exist and establish some properties of these limits. Subtracting (3.29) from (3.30) gives

$$\dot{f}_{ji} = -(\beta X_j + B_j)f_{ji} - \beta X_i f_{ii} \tag{3.31}$$

and by renaming indices

$$\dot{f}_{ii} = -(\beta X_i + B_i)f_{ii} - \beta X_j f_{ji} \tag{3.32}$$

where $f_{uv} = X_v - y_{uv}$, and $\{i, j\} = \{1, 2\}$. From (3.31) and (3.32) it is obvious by the positivity of βX_i and βX_j, and the argument used in Lemma 3.2 that if $f_{ji}(t_o) < 0$ and $f_{ii}(t_o) > 0$, then $f_{ji}(t) < 0$ and $f_{ii}(t) > 0$ for all $t \geq t_o$. Similarly, $f_{ji}(t_o) > 0$ and $f_{ii}(t_o) < 0$ implies $f_{ji}(t) > 0$ and $f_{ii}(t) < 0$ for all $t \geq t_o$. The same facts hold when the strict inequalities are replaced by weak inequalities. Moreover $f_{ii}(t_o) = f_{ji}(t_o) = 0$ implies $f_{ji}(t) = f_{ii}(t) = 0$ for all $t \geq t_o$. It is therefore obvious that the functions f_{ji} and f_{ii} change sign at most once, and not at all if $f_{ji}(0)f_{ii}(0) \leq 0$.

Also, $f_{ji}(t)$ and $f_{ii}(t)$ are identically zero if $f_{ji}(0) = f_{ii}(0) = 0$.

By (3.29) and the positivity of B_j, $\dot{y}_{ji}$ changes sign at most once and not at all if $f_{ji}(0) f_{ii}(0) \leq 0$. Thus y_{ji} is a monotonic function for large t. Since y_{ji} is also bounded and continuous, $P_{ji} = \lim_{t\to\infty} y_{ji}(t)$ exists.

Moreover $\dot{y}_{ji}(t) \equiv 0$ if $f_{ji}(0) = f_{ii}(0) = 0$. In this case $y_{ji}(t)$ is a constant, and since $f_{ji}(t) = f_{ii}(t) = 0$, $X_i(t) = y_{ji}(t) = y_{ii}(t) =$ constant. In particular, $Q_i = P_{ji} = P_{ii}$.

(III) We now use the fact that the limits P_{jk} exist to show that the limits Q_i exist. Since $X_1 + X_2 = 1$, it suffices to prove the existence of Q_1. Consider (3.28),

$$\dot{X}_1 = \beta((y_{11}-1)X_1 + (1-X_1)y_{21})$$

$$= \beta(-(y_{12} + y_{21})X_1 + y_{21}),$$

which has the integral form

$$X_1(t) = e^{-\int_0^t u_1(\xi)d\xi}\left[X_1(0) + \int_0^t \frac{y_{21}(v)}{y_{21}(v)+y_{12}(v)} \frac{d}{dv} e^{\int_0^v u_1(\xi)d\xi} dv\right],$$

where $U_1 = \beta(y_{12} + y_{21})$ and y_{21} are positive and have finite limits as $t\to\infty$. It is therefore obvious that Q_1 exists.

To show that the equations $Q_i P_{ij} = Q_j P_{ji}$ hold, note that (3.28) can be written as

$$\dot{X}_i = \beta[(y_{ii} - 1) X_i + X_j y_{ji}]$$

$$= \beta(-X_i y_{ij} + X_j y_{ji}).$$

Since all the limits Q_i and P_{jk} exist, the limit $\lim_{t\to\infty} \dot{X}_i(t)$ also exists and

equals $\beta(-Q_iP_{ij} + Q_jP_{ji})$. Since X_i is bounded, $\lim_{t\to\infty} \dot{X}_i(t) = 0$ and thus $Q_iP_{ij} = Q_jP_{ji}$.

(IV) We know from the case $X_1(0) = y_{11}(0) = y_{21}(0)$ and $X_2(0) = y_{22}(0) = y_{12}(0)$ that any probability distributions $Q_1 = P_{11} = P_{21}$ and $Q_2 = P_{22} = P_{12}$ can arise as limits. We now show that only limits Q_i and P_{jk} subject to these constraints can arise for $\sigma > 0$, and that these limits are positive if the initial data is positive. This we prove in two cases, in which we again always assume that all initial data is positive.

<u>Case 1.</u> Suppose $Q_1 = 0$. Then $Q_2 = 1$, and $0 = Q_1P_{12} = Q_2P_{21} = P_{21}$. Since $y_{21}(t)$ is a positive and monotonic function for large t and $P_{21} = 0$, $y_{21}(t)$ is a monotone <u>decreasing</u> function for large t. By (3.29) and the positivity of B_2, $y_{21}(t) \geq X_1(t)$ for large t. Consider (3.28). Then

$$\begin{aligned}\dot{X}_1 &= \beta(-X_1y_{12} + X_2y_{21}) \\ &= \beta(X_2(y_{21} - X_1) + X_1(X_2 - y_{12})) \\ &\geq \beta X_1(X_2 - y_{12}).\end{aligned}$$

Two possibilities arise. Either $X_2 - y_{12} \geq 0$ for all large t, or $X_2 - y_{12} \leq 0$ for all large t. In the former case, $\dot{X}_1 \geq 0$ for all large t. Since X_1 is positive, we conclude that $Q_1 > 0$, which is a contradiction. Suppose $X_2 - y_{12} \leq 0$ for all large t. Since $X_2 - y_{12} = 1 - X_1 - 1 + y_{11} = y_{11} - X_1$, then $y_{11} - X_1 \leq 0$ for all large t. By (3.29) $\dot{y}_{11} \geq 0$ and y_{11} is monotone increasing for all large t. Thus $y_{11} \leq X_1 \leq y_{21}$ for all large t, where y_{11} is a monotone increasing and positive function. Hence $P_{21} > 0$, which is a contradiction.

We have hereby shown that $Q_1 > 0$. Similarly $Q_2 > 0$.

<u>Case 2.</u> Suppose $Q_1 > 0$ and $Q_2 > 0$. Since $Q_1 > 0$,

$$\lim_{t\to\infty} B_1(t) = \lim_{t\to\infty} \frac{X_1(t)}{\gamma_1 e^{-\sigma t} + \int_0^t e^{-\sigma(t-\xi)} X_1(\xi)\,d\xi}$$

$$= \frac{Q_1}{Q_1/\sigma}$$

$$= \sigma$$

$$> 0$$

where

$$\gamma_1 = \frac{z^{(1)}(0)}{\beta x^2(0)} .$$

By (3.29), $\lim_{t\to\infty} \dot{y}_{12}(t)$ exists and equals $\sigma(Q_2 - P_{12})$. Since y_{12} is bounded, $\lim_{t\to\infty} \dot{y}_{12}(t) = 0$, and thus $Q_2 = P_{12}$. In a similar fashion we find $Q_2 = P_{22} = P_{12}$ and $Q_1 = P_{11} = P_{21}$, which concludes the proof.

The way in which the common limit $Q_i = P_{ii} = P_{ji}$ is approached by X_i, y_{ii}, and y_{ji} for large t can be spelled out very precisely. Exactly two kinds of alternatives exist in an interval $[T, \infty)$ if T is chosen sufficiently large. These alternatives are graphed in Figure 21.

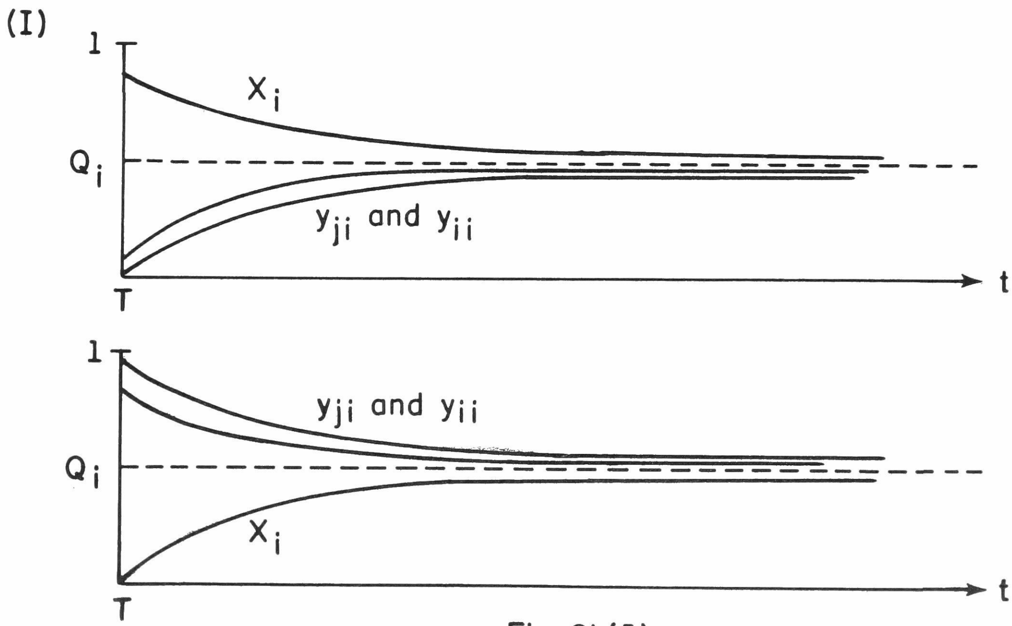


Fig. 21 (I)

(II)

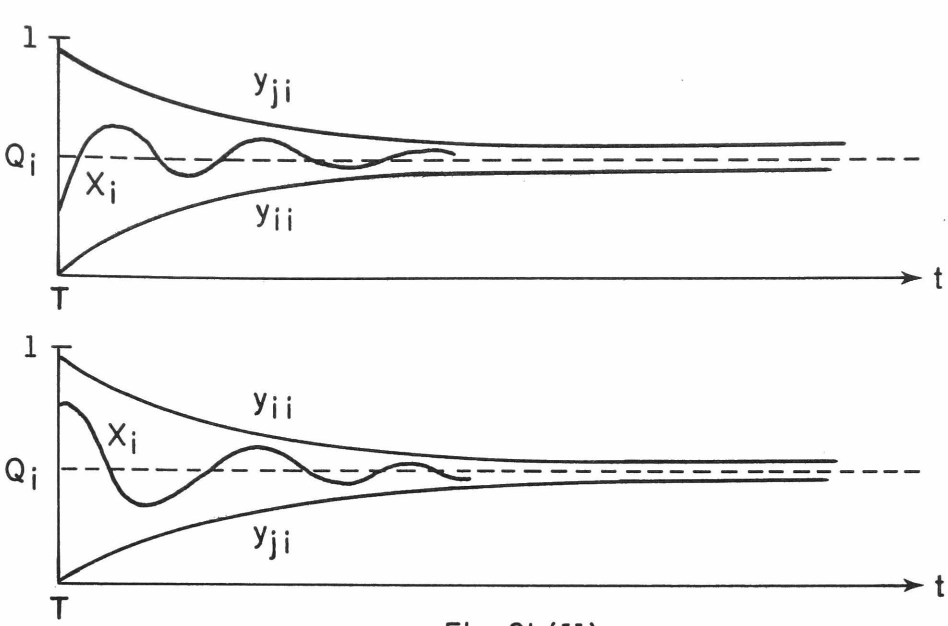


Fig. 21 (II)

Thus either X_i approaches its limit monotonically in the opposite sense from the monotonic approach of y_{ji} and y_{ii}, or the oscillations of X_i, if any, are pinched between y_{ji} and y_{ii} as they approach the common limit Q_i in opposite senses. These alternatives are proved in the followint corollary.

<u>COROLLARY 3.3.</u> One of the following alternatives holds for each triple (X_i, y_{ii}, y_{ii}), $\{i, j\} = \{1, 2\}$.

(1) $y_{ji} \geq X_i \geq y_{ii}$, y_{ji} is monotone decreasing, and y_{ii} is monotone increasing for all $t \geq 0$,

(2) $y_{ii} \geq X_i \geq y_{ji}$, y_{ii} is monotone decreasing, and y_{ji} is monotone increasing for all $t \geq 0$.

(3) $X_i \geq y_{ji}$, $X_i \geq y_{ii}$, y_{ji} and y_{ii} are monotone increasing, and X_i is monotone decreasing for all $t \geq 0$.

(4) $y_{ji} \geq X_i$, $y_{ii} \geq X_i$, y_{ji} and y_{ii} are monotone decreasing, and X_i is monotone increasing for all $t \geq 0$,

(5) Either (3) or (4) holds for small t, and becomes (1) or (2) for all large t.

In all cases, the common limit $Q_i = P_{ji} = P_{ii}$ lies within the interval $[m_i, M_i]$, where $m_i = \min\{X_i(0), y_{ji}(0), y_{ii}(0)\}$ and $M_i = \max\{X_i(0), y_{ji}(0), y_{ii}(0)\}$.

PROOF. (1) is a translation of two facts. Firstly $f_{ji}(0) \leq 0 \leq f_{ii}(0)$ implies $f_{ji}(t) \leq 0 \leq f_{ii}(t)$ for all $t \geq 0$. Secondly $\dot{y}_{ii} = B_i f_{ii} \geq 0$ and $\dot{y}_{ji} = B_j f_{ji} \leq 0$ for all $t \geq 0$. (2) is proved in a similar way. (3) says $f_{ji}(t) \geq 0$ and $f_{ii}(t) \geq 0$ for all $t \geq 0$. By (3.29), $\dot{X}_i = -\beta(X_i f_{ii} + X_j f_{ji}) \leq 0$ for all $t \geq 0$. (4) is the same situation as (3) with all inequalities reversed. Theorem 3.2 says that either (3) or (4) hold, or one of the functions $f_{ji}(t)$ and $f_{ii}(t)$ eventually changes sign. This is case (5).

The following corollary can be proved in the same way that Corollary 3.2 was proved.

COROLLARY 3.4. For arbitrary positive initial data and any σ,

$$|P_{jk} - y_{jk}(0)| \leq 2 \log\left(1 + \frac{1}{\gamma_j} \int_0^\infty e^{\sigma v} X_j \, dv\right),$$

where $\gamma_j = \dfrac{z^{(j)}(0)}{\beta X^2(0)} > 0.$

In particular, when $\sigma < 0$,

$$|P_{jk} - y_{jk}(0)| \leq 2 \log\left(1 + \frac{1}{\gamma_j |\sigma|}\right).$$

Thus taking $\sigma < 0$ and $|\sigma| >> 0$ forces P_{jk} to lie very close to $y_{jk}(0)$. We therefore find that a complete 2-graph with loops can remember its initial data both when $\sigma > 0$ and when $\sigma < 0$.

CHAPTER IV

A GLOBAL RATIO LIMIT THEOREM FOR GENERAL LINEARIZED COMPLETE GRAPHS WITHOUT LOOPS

1. GENERAL LINEARIZED COMPLETE GRAPHS WITHOUT LOOPS.

In the previous chapter we considered a special case of an input-free system whose coefficient matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \vdots & & & & \vdots \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & 0 \end{pmatrix}.$$

The general input-free system with this coefficient matrix P is

$$\begin{aligned} \dot{x}_i(t) &= -\alpha x_i(t) + \beta \textstyle\sum_{k=1}^{n} x_k(t-\tau)\, y_{ki}(t), \quad i = 1,2,\ldots,n \\ y_{ki}(t) &= \frac{z_{ki}(t)}{\sum_{j=1}^{n} z_{kj}(t)}, \\ \dot{z}_{ki}(t) &= -u z_{ki}(t) + \beta x_k(t-\tau)\, x_i(t), \quad k \neq i \\ z_{ii}(t) &= 0. \end{aligned} \qquad (*)$$

The system of Theorem 3.1 is characterized by the choices $n=3$ and $\tau=0$. In this case, we observed that the ratios $X_i(t)$ and $y_{jk}(t)$ of a system with positive $\alpha, \beta,$ and $u,$ and positive locally reversible initial data always have limits Q_i and P_{jk} as $t \to \infty$. If moreover $\sigma \equiv u+2(\beta-\alpha)>0$, these limits are unique and equal $Q_i = \frac{1}{3}$ and $P_{jk} = \frac{1}{2}(1-\delta_{jk})$. That is, the variables $x_i(t)$ and $z_{jk}(t)$ become independent of their indices $i, j,$ and $k,$ $j \neq k,$ for large t. Any solution of (*) of the form $x_i(t)=\gamma(t)$ and $z_{jk}(t)=\delta(t)(1-\delta_{jk})$ is called a <u>uniform solution</u> of (*). $\sigma > 0$ thus implies that a uniform solution is approached as $t \to \infty$.

Each system (*) gives rise to an infinite set of linearized systems in a manner which will be described in detail in Section 3. Each of these linearized systems has the same number n of vertices as (*), the same lag time τ, and describes a linearized flow over a graph with the same coefficient matrix P as (*). These linearized systems arise, as in Theorem 3.1, from an interest in the behavior for large t of the solutions of (*) whose initial data has the following properties: (1) $z_{jk}(0)>0$, $j\neq k$, and (2) $x_i(\nu)$ is a continuous and positive function of $\nu\in[-\tau,0]$, $i=1,2,\ldots,n$. Such a solution is called a positive solution of (*) since by Theorem 1.1, $z_{jk}(t)>0$, $j\neq k$ and $x_i(t)>0$, $i=1,2,\ldots,n$ for all $t\geq 0$. In Theorem 3.1 an arbitrary positive solution with locally reversible initial data approaches a positive uniform solution as $t\to\infty$ if $\sigma>0$. In this chapter, we therefore compare, in first approximation, any positive solution of (*) with a properly chosen positive uniform solution of (*) as $t\to\infty$. This comparison is expressed formally as a ratio limit theorem for the linearized systems.

We will prove ratio limit theorems for linearized systems with arbitrarily many vertices $(n\geq 3)$ and with an arbitrary lag time $(\tau\geq 0)$. Our goal is to see how we must change the condition $\sigma\equiv u+2(\beta-\alpha)>0$ to guarantee that a uniform solution is approached as $t\to\infty$ in linearized systems with $n>3$ or $\tau>0$. The following results hold.

In all the linearized systems which we will treat, we will find a sufficient condition on n and τ, for fixed α,β, and u, which guarantees convergence to a uniform solution for a wide class of initial data. For fixed τ, this condition becomes weaker as the number of vertices n becomes larger. We therefore say that stability properties of the linearized system are graded in n. This condition is automatically fulfilled for all $n\geq 3$ when $\tau=0$ if $\sigma>0$, which is compatible with the result of Theorem 3.1. Moreover, if convergence to a uniform solution holds for a linearized system with a fixed number n of vertices and a fixed $\tau=\tau_0$, then it also holds for all systems with n vertices and τ lying in a suitably chosen neighborhood containing τ_0. If also $u>2(\alpha-\beta)>0$ (i.e., $\lim_{t\to\infty} x_i(t)=0$ for all $\tau\geq 0$), then for fixed n,

a uniform solution is approached for all $T\in[0,\omega(n)]$, where $\omega(n)$ is monotone increasing in n and $\lim_{n\to\infty}\omega(n)=\infty$.

Before carrying out the linearized comparison between positive solutions and positive uniform solutions of (*), we summarize several properties of the positive uniform solutions of (*).

2. POSITIVE UNIFORM SOLUTIONS.

The equation

$$\dot{x}(t) = -\alpha x(t) + \beta x(t-\tau)$$

is obeyed by the average output $x=\frac{1}{n}\sum_{k=1}^{n} x_k$ of (*). Given any positive uniform solution of (*), where we suppose $x_i=\gamma>0$ and $z_{ij}=\delta(1-\delta_{ij})$, $\delta>0$, it is obvious that

$$\dot{\gamma}(t) = -\alpha\gamma(t) + \beta\gamma(t-\tau), \qquad (4.1)$$

since $x_i=x$. Since $\delta=z_{ij}$ for all $i\neq j$

$$\dot{\delta}(t) = -u\delta(t) + \beta\gamma(t-\tau)\gamma(t). \qquad (4.2)$$

Equations (4.1) and (4.2) completely define a positive uniform solution of (*) once its initial data is specified. (4.1) has been thoroughly studied in the mathematical literature ([3], [9]). In the present account, we merely list the known facts we shall need, and derive several simple new facts from them.

We shall actually need some results for solutions of (4.1) and (4.2) which are not necessarily positive, but which include positive solutions as a special case. These results can be stated in terms of the following definitions. Given any $T>0$ and any $f\in C[0,T]$, let

$$K_\tau(f) \equiv f(\tau) + \beta \int_0^\tau f(v) e^{-vs(\tau)} dv, \tag{4.3}$$

where $s(\tau)$ is the largest real part of the roots of the equation

$$P_\tau(s) \equiv s + \alpha - \beta e^{-\tau s} = 0 \tag{4.4}$$

Also let $\sigma(\tau) \equiv u + 2s(\tau)$.

<u>PROPOSITION 4.1</u> Let ξ and η be any solutions of the differential equations

$$\dot{\xi}(t) = -\alpha\xi(t) + \beta\xi(t-\tau) \tag{4.5}$$

and

$$\dot{\eta}(t) = -u\eta(t) + \beta\xi(t-\tau)\xi(t) \tag{4.6}$$

whose initial data is chosen such that $\xi(v)$ is a continuously differentiable function of $v \in [0, \tau]$ with $K_\tau(\xi) \neq 0$. Then

$$\lim_{t\to\infty} \frac{\xi(t-\tau)}{\xi(t)} = e^{-\tau s(\tau)}, \tag{4.7}$$

$$\lim_{t\to\infty} \frac{d}{dt} \frac{\xi(t-\tau)}{\xi(t)} = 0, \tag{4.8}$$

and

$$\lim_{t\to\infty} \frac{\xi^2(t-\tau)}{\eta(t)} = \frac{1}{\beta}\sigma(\tau) e^{-\tau s(\tau)}. \tag{4.9}$$

If ξ_1 and ξ_2 are any two solutions of (4.5) chosen in this way, and η_1 is a solution of

$$\dot{\eta}_1(t) = -u\eta_1(t) + \beta\xi_1(t-\tau)\xi_1(t),$$

then

$$\lim_{t\to\infty} \frac{d}{dt} \log \frac{\xi_2(t)\eta_1(t)}{\xi_1(t-\tau)} = \sigma(\tau) - u. \tag{4.10}$$

Moreover, there exists a t_0 such that

$$\int_{t_0}^{\infty} \left| \frac{\xi(s-\tau)}{\xi(s)} - e^{-\tau s(\tau)} \right| ds < \infty , \tag{4.11}$$

$$\int_{t_0}^{\infty} \left| \frac{d}{ds} \frac{\xi(s-\tau)}{\xi(s)} \right| ds < \infty , \tag{4.12}$$

$$\int_{t_0}^{\infty} \left| \frac{\varepsilon^2(s-\tau)}{\eta(s)} - \frac{1}{\beta}\sigma(\tau) e^{-\tau s(\tau)} \right| ds < \infty , \tag{4.13}$$

and

$$\int_{t_0}^{\infty} \left| \frac{d}{ds} \log \frac{\xi_2(s)\eta_1(s)}{\xi_1(s-\tau)} + u - \sigma(\tau) \right| ds < \infty . \tag{4.14}$$

If the initial data of (4.5) is merely continuous in $[-T, 0]$, then the solution of (4.5) is continuously differentiable in $[0, T]$, so that differentiability in $[0, T]$ is not a restrictive assumption. The proof of these facts relies upon the following lemma.

LEMMA 4.1 For any fixed $T > 0$, the root $s_1(T)$ of largest real part of the equation

$$P_\tau(s) \equiv s + \alpha - \beta e^{-\tau s} = 0 \tag{4.4}$$

is real. Thus $s(T) = s_1(T)$. Moreover, only finitely many roots of this equation have a nonnegative real part.

PROOF. Suppose that $s = x + iy$ is a root of (4.4). Then

$$x + \alpha - \beta e^{-\tau x} \cos \tau y = 0 \tag{4.15}$$

and

$$y + \beta e^{-\tau x} \sin \tau y = 0 . \tag{4.16}$$

Write (4.15) in the form $y(x) = z_0(x)$, where $y(x) = x$, $z_0(x) = -\alpha + \beta \Theta e^{-Tx}$, and

$\theta = \cos \tau y \in [-1, 1]$. For each fixed $\theta \in [-1, 1]$, we consider the graphs of $y(x)$ and $z_\theta(x)$ as functions of x. Every root of (4.15) must lie at the intersection of these graphs for some $\theta \in [-1, 1]$. For example, if $\alpha > \beta > 0$ we find Figure 22

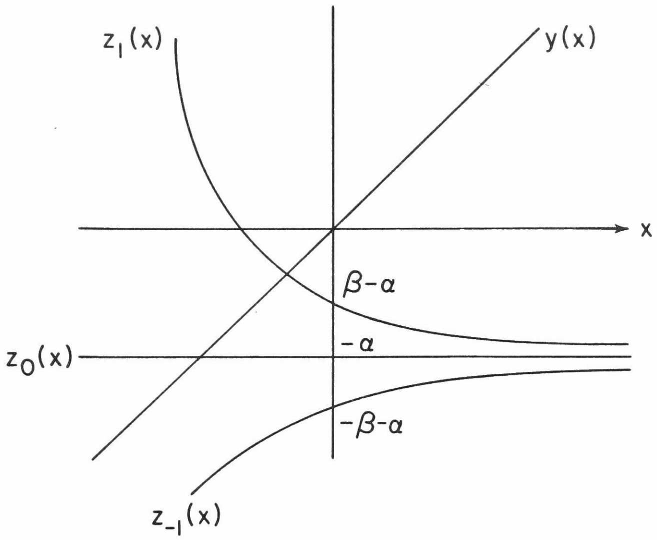


Fig. 22

It is clear from this diagram that the root x_1 of $y(x) = z_1(x)$ is a simple root and is the root with largest real part among all roots of the equations $y(x) = z_\theta(x)$, $\theta \in [-1, 1]$. When $\theta = 1$, $\cos \tau y = 1$ and $\sin \tau y = 0$. Thus by (4.16), the imaginary part y_1 corresponding to x_1 is $y_1 = -\beta e^{-\tau x} \sin \tau y_1 = 0$. The zero of largest real part of $P_\tau(s)$ is therefore a real and simple zero of $P_\tau(s)$.

Since $P_\tau(s)$ is a nontrivial entire function of s, only finitely many zeros of $P_\tau(s)$ can occur in any finite region of the s plane. For any zero $s_k = x_k + iy_k$ with nonnegative real part x_k, we have by (4.16) the inequality $|y_k| \le \beta e^{-\tau x_k} |\sin \tau y_k| \le \beta$. Thus at most finitely many zeros of $P_\tau(s)$ have a nonnegative real part.

We can now prove Proposition 4.1. We follow [3], p. 109, for much of the proof. Let $e^{s_k t} p_k(t)$ denote the residue of the function $e^{-ts}[P_\tau(s)]^{-1} p(s)$ at a zero s_k of $P_\tau(s)$, where $P_\tau(s)$ is defined in (4.4) and

$$p(s) = \xi(\tau) e^{-\tau s} + (s + \alpha) \int_0^\tau \xi(v) e^{-vs}\, dv.$$

Let $\{s_k\}$ be the sequence of zeros of $P_\tau(s)$ arranged in order of decreasing real parts. Then the solution ξ of (4.5) can be written as the infinite series

$$\xi(t) = \sum_{k=1}^{\infty} p_k(t) e^{s_k t} \qquad \text{for} \qquad t > \tau.$$

This series converges uniformly for t in any finite interval $[t_0, t_0']$ where $t_0 > T$. Moreover if $\operatorname{Re} s_k \leq c < 0$ for all $k = 1, 2, \ldots$, then the series converges uniformly for $t \in [t_0, \infty]$, where $t_0 > T$. At most one zero of $P_T(s)$ is not a simple zero. We shall treat explicitly only the case in which all zeros are simple. This case arises whenever $1 \neq \beta T \exp(1 + \alpha T)$. Our results carry through to the general case, since the nonsimple zero is readily seen not to be the zero with largest real part.

Suppose all zeros of $P_T(s)$ are simple. Then the residue of $e^{ts}[P_T(s)]^{-1} p(s)$ at s_k is $e^{ts_k} \dfrac{p(s_k)}{P_T'(s_k)}$, and

$$\xi(t) = \sum_{k=1}^{\infty} c_k e^{s_k t} \tag{4.17}$$

where $c_k = \dfrac{p(s_k)}{P_T'(s_k)}$. c_k can be written in a simplied form as follows. Since

$$P_\tau(s_k) = s_k + \alpha - \beta e^{-\tau s_k} = 0,$$

$p(s_k)$ can be written in the form

$$p(s_k) = e^{-\tau s_k}\left[\xi(\tau) + \beta \int_0^\tau \xi(v) e^{-s_k v} dv\right].$$

Noting also that $P_T'(s) = 1 + \beta T e^{-Ts}$, we find

$$c_k = \frac{e^{-\tau s_k}\left[\xi(\tau) + \beta \int_0^\tau \xi(v) e^{-s_k v} dv\right]}{1 + \beta \tau e^{-\tau s_k}}. \tag{4.18}$$

By Lemma 4.1, the zero s_1 of largest real part of $P_T(s)$ is real (i.e., $s(T) = s_1$), and only finitely many roots have nonnegative real parts. We

can summarize these facts by writing the series (4.17) in the form

$$\xi(t) = e^{s(\tau)t}\left(c_1 + F(t) + G(t)\right) \tag{4.19}$$

where $e^{s(T)t}F(t)$ is the finite sum

$$\sum_{k=2}^{m} c_k e^{s_k t}$$

over the terms $c_k e^{s_k t}$, $k \geq 2$, with $\operatorname{Re} s_k \geq 0$ and $e^{s(T)t}G(t)$ is the infinite sum

$$\sum_{k=m+1}^{\infty} c_k e^{s_k t}$$

over the terms $c_k e^{s_k t}$ with $\operatorname{Re} s_k < 0$. Since $s(T) > \operatorname{Re} s_k$, $k > 1$, each of the summands in

$$F(t) = \sum_{k=2}^{m} c_k e^{(s_k - s(\tau))t}$$

and in

$$G(t) = \sum_{k=m+1}^{\infty} c_k e^{(s_k - s(\tau))t}$$

has a negative real part. We shall use this fact to write (4.19) in the following form

$$\xi(t) = e^{s(\tau)t}\left(c_1 + e^{-\omega t} M(t)\right), \tag{4.20}$$

where

$$c_1 = \frac{e^{-\tau s(\tau)} K_\tau(\xi)}{1 + \beta\tau e^{-\tau s(\tau)}} \neq 0,$$

$\omega > 0$, and $M(t)$ is a bounded function.

(4.20) can be proved by writing $F(t)$ and $G(t)$ separately as a product of an exponentially decreasing term and a bounded function and then adding.

$$F(t) = e^{(x_2 - s(\tau))t} N(t)$$

where

$$N(t) = \sum_{k=2}^{m} c_k e^{(s_k - x_2)t}$$

is obviously bounded. Using $F(t)$ as a guide, write $G(t)$ as

$$G(t) = e^{(x_2 - s(\tau))t} Q(t)$$

where

$$Q(t) = \sum_{k=m+1}^{\infty} c_k e^{(s_k - x_2)t} .$$

It remains only to show that $Q(t)$ is bounded. This fact is an immediate consequence of the following asymptotic formula for the roots $s=x+iy$ of $P_T(s)=0$ ([3], p. 416)

$$x = \frac{1}{\tau} \log \frac{\beta\tau}{2k\pi} + o(1),$$

and

$$y = \frac{\pi}{\tau}\left(2k - \frac{1}{2}\right) + o(1)$$

where k is any large integer. e^{xt} therefore has the asymptotic form

$$e^{xt} = \left[e^{o(1)}\left(\frac{\beta\tau}{2\pi k}\right)\right]^{\frac{t}{\tau}}$$

For sufficiently large t, $Q(t)$ can be shown to be bounded by comparison with the series $\sum_k \frac{1}{k^2}$.

From (4, 20), it is obvious that

$$\lim_{t\to\infty} \frac{\xi(t-\tau)}{\xi(t)} = e^{-\tau s(\tau)} . \tag{4. 7}$$

The limit

$$\lim_{t\to\infty} \frac{d}{dt} \frac{\xi(t-\tau)}{\xi(t)} = 0 \tag{4. 8}$$

follows from (4. 7) and the observation that

$$\frac{\dot{\xi}(t)}{\xi(t)} = -\alpha + \frac{\beta\xi(t-\tau)}{\xi(t)} ,$$

since then

$$\begin{aligned}\lim_{t\to\infty} \frac{d}{dt} \frac{\xi(t-\tau)}{\xi(t)} &= \lim_{t\to\infty} \frac{\xi(t-\tau)}{\xi(t)} \lim_{t\to\infty} \left(\frac{\dot{\xi}(t-\tau)}{\xi(t-\tau)} - \frac{\dot{\xi}(t)}{\xi(t)}\right) \\ &= e^{-\tau s(\tau)} \cdot 0 \\ &= 0 .\end{aligned}$$

We can now prove (4. 9). By (4. 6), we have for all $t \geq T$,

$$\eta(t) = e^{-ut}\left(\eta(\tau)e^{u\tau} + \beta\int_\tau^t e^{uv}\xi(v-\tau)\xi(v)dv\right) . \tag{4. 21}$$

The integral in (4. 15) can be estimated using the equality

$$\xi(v-\tau)\xi(v) = e^{2s(\tau)v} e^{-\tau s(\tau)}\left[c_1^2 + e^{-\omega v}R(v)\right]$$

where $R(v)$ is bounded. Thus

$$\eta(t) = e^{-ut}\left[\eta(\tau)e^{u\tau} + \beta c_1^2 e^{-\tau s(\tau)}\left(\frac{e^{\sigma(\tau)t} - e^{\sigma(\tau)\tau}}{\sigma(\tau)}\right) + \int_\tau^t e^{(\sigma(\tau)-\omega)v} R(v)\,dv\right],$$

and

$$\eta(t) = e^{-ut}\left[\frac{\beta c_1^2 e^{-\tau s(\tau)} e^{\sigma(\tau)t}}{\sigma(\tau)} + O(1)\left(1 + e^{(\sigma(\tau)-\omega)t}\right)\right]. \tag{4.22}$$

Using (4.22) in conjunction with the equation

$$\xi^2(t-\tau) = e^{2s(\tau)t} e^{-2\tau s(\tau)}\left[c_1^2 + e^{-\omega t}R_1(t)\right],$$

where $R_1(t)$ is a bounded function, we readily find that

$$\lim_{t\to\infty} \frac{\xi^2(t-\tau)}{\eta(t)} = \frac{1}{\beta}\sigma(\tau)e^{-\tau s(\tau)}. \tag{4.9}$$

We now prove (4.10). By (4.7) and (4.9),

$$\lim_{t\to\infty} \frac{\xi_1(t-\tau)\xi_1(t)}{\eta_1(t)} = \lim_{t\to\infty} \frac{\xi_1^2(t-\tau)}{\eta_1(t)} \lim_{t\to\infty} \frac{\xi_1(t)}{\xi_1(t-\tau)}$$

$$= \frac{1}{\beta}\sigma(\tau).$$

Thus

$$\lim_{t\to\infty} \frac{d}{dt} \log \frac{\xi_2(t)\eta_1(t)}{\xi_1(t-\tau)}$$

$$= \lim_{t\to\infty} \left(\frac{\dot{\xi}_2(t)}{\xi_2(t)} + \frac{\dot{\eta}_1(t)}{\eta_1(t)} - \frac{\dot{\xi}_1(t-\tau)}{\xi_1(t-\tau)} \right)$$

$$= \lim_{t\to\infty} \left(-\alpha + \frac{\beta\xi_2(t-\tau)}{\xi_2(t)} - u + \frac{\beta\xi_1(t-\tau)\xi_1(t)}{\eta_1(t)} + \alpha - \frac{\beta\xi_1(t-2\tau)}{\xi_1(t-\tau)} \right)$$

$$= -u + \sigma(\tau). \tag{4.10}$$

To prove (4.11), note that for t sufficiently large,

$$\frac{\xi(t-\tau)}{\xi(t)} - e^{-\tau s(\tau)} = \frac{e^{s(\tau)(t-\tau)}\left(c_1 + e^{-\omega(t-\tau)}M(t-\tau)\right)}{e^{s(\tau)t}\left(c_1 + e^{-\omega t}M(t)\right)} - e^{-\tau s(\tau)}$$

$$= e^{-\tau s(\tau)}\left(1 + e^{-\omega t}N(t)\right) - e^{-\tau s(\tau)}$$

$$= e^{-\omega t} e^{-\tau s(\tau)} N(t),$$

where $N(t)$ is a bounded function. (4.11) follows immediately. The other inequalities (4.12), (4.13), and (4.14) can be proved in an identical way. With these limiting relations in mind, we now turn to a comparison of positive solutions (*) and positive uniform solutions of (*).

3. THE VARIATIONAL SYSTEM.

For convenience in the following discussion, we write (*) in matrix form as

$$\dot{u}(t) = f(u(t), u(t-\tau)) \qquad (*)$$

in terms of the n^2 dimensional vector $U=(x_1, \ldots, x_n, z_{12}, z_{13}, \ldots, z_{n,n-2}, z_{n,n-1})$ and the n^2 dimensional vector $f=(f_1, \ldots, f_n, f_{12}, f_{13}, \ldots, f_{n,n-2}, f_{n,n-1})$ where

$$f_i = -\alpha x_i + \beta \Sigma_{k=1}^{n} x_k(t-\tau) z_{ki} \left(\Sigma_{j \neq k} z_{kj}\right)^{-1}$$

and

$$f_{jk} = -u z_{jk} + \beta x_j(t-\tau) x_k, \qquad j \neq k$$

We shall linearize (*) in the following way. Our linearization is motivated by the desire to compare any positive solution $U=(x_1, \ldots, x_n, z_{12}, \ldots, z_{n,n-1})$ of (*) with a properly chosen positive uniform solution $U_0=(\gamma, \ldots, \gamma, \delta, \ldots, \delta)$ of (*) to generalize the result of Theorem 3.1 that

$$\frac{x_i(t)}{\Sigma_{k=1}^{3} x_k(t)} - \frac{1}{3} \longrightarrow 0 \quad \text{as} \quad t \to \infty, \quad i=1,2,3$$

to

$$\frac{x_i(t)}{\Sigma_{k=1}^{n} x_k(t)} - \frac{1}{n} \longrightarrow 0 \quad \text{as } t \to \infty, \quad i=1,2,\ldots,n. \tag{4.23}$$

This would be accomplished if it could be shown that $V \equiv U-U_0 \equiv (v_1, \ldots, v_n, v_{12}, \ldots, v_{n,n-1})$ satisfies

$$\frac{v_i(t)}{\Sigma_{k=1}^{n} v_k(t)} - \frac{1}{n} \longrightarrow 0 \quad \text{as } t \to \infty, \quad i=1,2,\ldots,n,$$

since if γ is chosen equal to $x=\Sigma_{k=1}^{n} x_k$, as can be done because both γ and x are positive solutions of (4.5), then

$$\frac{V_i}{\sum_{k=1}^{n} V_k} - \frac{1}{n} = \frac{1}{1-n}\left(\frac{X_i}{\sum_{k=1}^{n} X_k} - \frac{1}{n}\right).$$

An equation for V can be derived formally by simply expanding in a Taylor's series as follows.

$$\dot{V}(t) = \dot{u}(t) - \dot{u}_0(t) = f(u(t), u(t-\tau)) - f(u_0(t), u_0(t-\tau))$$

$$= f(u_0(t) + V(t), u_0(t-\tau) + V_0(t-\tau)) - f(u_0(t), u_0(t-\tau))$$

$$= f_\xi(u_0(t), u_0(t-\tau))V(t) + f_\eta(u_0(t), u_0(t-\tau))V(t-\tau)$$

$$+ o(\|V\|). \qquad (**)$$

$f_\xi(U_0(t), U_0(t-T)) = f_\xi(\xi,\eta)\big|_{(\xi,\eta)=(U_0(t), U_0(t-T))}$ is the matrix of partial derivatives of $f(\xi,\eta)$ taken with respect to the entries in the vector ξ and evaluated at $(\xi,\eta)=(U_0(t), U_0(t-T))$; $f_\eta(U_0(t), U_0(t-T)) = f_\eta(\xi,\eta)\big|_{(\xi,\eta)=(U_0(t), U_0(t-T))}$ is the matrix of partial derivatives of $f(\xi,\eta)$ taken with respect to the entries in the vector η and evaluated at $(\xi,\eta)=(U_0(t), U_0(t-T)$; $\|V\| = \sup_{\omega\in[t,t-T]} |V(\omega)|$.

Equation $(**)$ is discussed in [12], Chapter 4. In particular, when $\alpha > \beta$,

$$u(t) = O\left(e^{(\beta-\alpha)t}\right)$$

and

$$u_0(t) = O\left(e^{(\beta-\alpha)t}\right)$$

so that

$$V(t) = O\left(e^{(\beta-\alpha)t}\right)$$

and the higher order correction terms $o(\|V\|)$ are exponentially small for all large t. We will study $(**)$ when all higher order terms $o(\|V\|)$ are ignored. We therefore consider the infinite collection of linear systems

$$\dot{W}(t) = f_\xi(u_0(t), u_0(t-\tau))W(t) + f_\eta(u_0(t), u_0(t-\tau))W(t-\tau), \qquad (\ddagger)$$

where U_0 is any positive uniform solution of (✱). (‡) is called the <u>variational system</u> of (✱) ([12], p. 341).

We now study how the ratios of solutions of the variational system (‡) depend on the functions $k(t)=\beta e^{-Ts(T)}$ and $\sigma(T)=u+2s(T)$ of $T\geq 0$.

<u>THEOREM 4.1.</u> Let the numerical coefficients of (‡) be chosen to satisfy $\beta>0$, $\sigma(T)>0$, and $k(T)+\sigma(T)>\frac{1}{n-1}k(T)(1+T\sigma(T))$. Let $U_0(t)$ be any positive uniform solution of (✱) and, for fixed $U_0(t)$, let

$$W(t)=(h_1,\ldots,h_n,h_{12},\ldots,h_{n,n-1})$$

be any solution of the variational system (‡) whose initial data satisfies $K_T(\Sigma_{i=1}^n h_i)\neq 0$. Then there exist positive constants ω_1 and ω_2 such that

$$\frac{h_i(t)}{\Sigma_{k=1}^n h_k(t)}-\frac{1}{n}=o\left(e^{-\omega_1 t}\right), \tag{4.24}$$

and

$$\frac{h_{jk}(t)}{\Sigma_{\substack{m=1\\ m\neq j}}^n h_{jm}(t)}-\frac{1}{n-1}=o\left(e^{-\omega_2 t}\right). \tag{4.25}$$

Theorem 4.1 says that ratios of solutions of the variational system for which $K_T(\Sigma_{i=1}^n h_i)\neq 0$ approach a uniform solution at an exponential rate if $\beta>0$, $\sigma(T)>0$ and $k(T)+\sigma(T)>\frac{1}{n-1}k(T)(1+T\sigma(T))$. Equation (4.24) is obviously the linearized version of (4.23), and (4.25) is the linearized version of the limiting relation $y_{jk}(t)-\frac{1}{n-1}\to\infty$. The condition $K_T(\Sigma_{i=1}^n h_i)\neq 0$ is automatically fulfilled if the initial data of $W(t)$ is chosen such that $\Sigma_{i=1}^n h_i(v)=(1-n)\gamma(v)>0$ for $v\in[-T,0]$, as we did for $V(t)$ in the paragraph following (4.23). The condition $k(T)+\sigma(T)>\frac{1}{n-1}k(T)(1+T\sigma(T))$ is automatically fulfilled for all $n\geq 3$ when $T=0$ if $\sigma(0)>0$ since it becomes $\beta+\sigma(0)\geq\frac{1}{n-1}\beta$. We shall show in a Corollary to Theorem 4.1 that $\sigma(0)=\sigma\equiv u+2(\beta-\alpha)$. Thus the conditions on $k(T)$ and $\sigma(T)$ are automatically fulfilled when $T=0$ if $\sigma>0$, as was required in Theorem 3.1.

PROOF. The proof is divided into six steps. Step (I) consists merely in writing out the variational system ($\ddagger$) in terms of its components h_i and h_{jk}. These components obey the equations

$$\dot{h}_i = -\alpha h_i + \frac{\beta\gamma(t-\tau)}{(n-1)^2\delta(t)}\left[(n-1)H_i - H + H^{(i)}\right] + \frac{\beta}{n-1} h^{(i)}(t-\tau), \tag{4.26}$$

$$i = 1, 2, \ldots, n,$$

where

$$H_i = \Sigma_{j\neq i}\, h_{ji}, \quad H^{(i)} = \Sigma_{j\neq i}\, h_{ij};$$

$$H = \Sigma_{\substack{i,j \\ i\neq j}}\, h_{ij}, \text{ and } h^{(i)} = \Sigma_{j\neq i}\, h_j;$$

and

$$\dot{h}_{jk} = \beta\gamma(t) h_j(t-\tau) + \beta\gamma(t-\tau) h_k - u h_{jk}, \quad j\neq k. \tag{4.27}$$

Step (II) shows that the sum $h = \Sigma_{k=1}^{n} h_k$ of the solutions of (4.26) obeys the equation

$$\dot{h}(t) = -\alpha h(t) + \beta h(t-\tau), \tag{4.28}$$

and that the sum $H = \Sigma_{\substack{i,j \\ i\neq j}} h_{ij}$ of the solutions of (4.27) obeys the equation

$$\dot{H}(t) = -uH(t) + \beta(n-1)\left(\gamma(t) h(t-\tau) + \gamma(t-\tau) h(t)\right). \tag{4.29}$$

In step (III), we use (4.28) to simplify (4.26) in the following way. Each of the n equations in (4.26) has a right hand side which depends on all n^2

variables h_i and h_{jk} . We shall nonetheless be able to transform the i^{th} equation into an equation in which only one unknown function appears, namely the function $g_i = \frac{h_i}{\Sigma_{k=1}^{n} h_k} - \frac{1}{n}$, A price is paid for this simplification. g_i obeys a pair of <u>coupled</u> difference-differential equations, namely

$$\dot{g}_i = -\beta \frac{h(t-\tau)}{h(t)}\left(g_i + \frac{1}{n-1} g_i(t-\tau)\right) + \frac{\beta\gamma(t-\tau)G_i}{(n-1)^2 h\,\delta}, \tag{4.30}$$

where

$$\dot{G}_i = -uG_i + \beta n(n-2)\gamma(t-\tau) h\, g_i . \tag{4.31}$$

Equations (4.30) and (4.31) may be thought of as an "uncoupling" of the variables h_i and h_{jk} in (4.26). In step (IV), we differentiate (4.30) and use (4.31) to find a second-order equation for g_i of the form

$$\ddot{g}_i + A(t)\dot{g}_i + B(t)\dot{g}_i(t-\tau) + C(t) g_i + D(t) g_i(t-\tau) = 0 . \tag{4.32}$$

Such an equation, for general variable coefficients $A(t)$, $B(t)$, $C(t)$, and $D(t)$ would provide us with little information about g_i as $t \to \infty$. It is fortunate that in the present situation these coefficients are constructed from algebraic combinations of the functions $\frac{h(t-T)}{h(t)}$, $\frac{d}{dt}\frac{h(t-T)}{h(t)}$, $\frac{\gamma^2(t-T)}{\delta(t)}$, and $\frac{d}{dt}\log\frac{h(t)\delta(t)}{\gamma(t-T)}$. Thus by (4.7), (4.8), (4.9), and (4.10) the limits $A = \lim_{t\to\infty} A(t)$, $B = \lim_{t\to\infty} B(t)$, $C = \lim_{t\to\infty} C(t)$, and $D = \lim_{t\to\infty} D(t)$ exist. We can therefore compare the behavior of (4.32) for large t with the behavior of the solution w_i of the following equation with <u>constant</u> coefficients.

$$\ddot{w}_i + A\dot{w}_i + B\dot{w}_i(t-\tau) + Cw_i + Dw_i(t-\tau) = 0 . \tag{4.33}$$

It will actually be more useful to make a change of variables in (4.32) and (4.33) to $\xi_i = g_i e^{\lambda t}$ and $\eta_i = w_i e^{\lambda t}$ where λ is a sufficiently small positive constant. In terms of the new variables ξ_i and η_i, we find equations of the form

$$\ddot{\xi}_i + \bar{A}(t)\dot{\xi}_i + \bar{B}(t)\dot{\xi}_i(t-\tau) + \bar{C}(t)\xi_i + \bar{D}(t)\xi_i(t-\tau) = 0, \tag{4.34}$$

and

$$\ddot{\eta}_i + \bar{A}\dot{\eta}_i + \bar{B}\dot{\eta}_i(t-\tau) + \bar{C}\eta_i + \bar{D}\eta_i(t-\tau) = 0. \tag{4.35}$$

We compare equations (4.34) and (4.35) to show that ξ_i is bounded whenever η_i is bounded, and in (V) we show that η_i actually converges to zero as $t \to \infty$ because all zeros of the exponential polynomial

$$G^{(\lambda)}_{\beta n \tau}(s) = s^2 + \bar{A}s + (\bar{B}s + \bar{D})e^{-\tau s} + \bar{C}$$

have negative real parts. From these facts it readily follows that

$$g_i \equiv \frac{h_i}{\sum_{k=1}^n h_k} - \frac{1}{n} = o\left(e^{-\lambda t}\right),$$

which is the first conclusion of the theorem. In step (VI), we use the conclusion of step (V) to imitate the method used on (4.26) as far as possible on (4.27). From these manipulations, the conclusion that

$$\frac{h_{jk}}{\sum_{\substack{m=1 \\ m \neq j}}^n h_{jm}} - \frac{1}{n-1}$$

converges exponentially to zero will readily follow.

(I) <u>The Variational System in Component Form.</u> We now write out the variational system

$$\dot{W}(t) = f_\xi(U_0(t), U_0(t-\tau))W(t) + f_\eta(U_0(t), U_0(t-\tau))W(t-\tau) \quad (\ddagger)$$

in terms of the components h_i and h_{jk} of $W=(h_1, \ldots, h_n, h_{12}, \ldots, h_{n,n-1})$, where the vector function $f(\xi,\eta)=(f_1(\xi,\eta), \ldots, f_n(\xi,\eta), f_{12}(\xi,\eta), \ldots, f_{n,n-1}(\xi,\eta))$ of $\xi=(\xi_1, \ldots, \xi_n, \xi_{12}, \ldots, \xi_{n,n-1})$ and $\eta=(\eta_1, \ldots, \eta_n, \eta_{12}, \ldots, \eta_{n,n-1})$ is given by

$$f_i(\xi,\eta) = -\alpha\xi_i + \beta\sum_{k\neq i}\eta_k\xi_{ki}\left(\sum_{j\neq k}\xi_{kj}\right)^{-1}$$

and

$$f_{jk}(\xi,\eta) = -u\xi_{jk} + \beta\eta_j\xi_k.$$

Since the computation needed to do this is straightforward but tedious, we merely give two examples of how the uniformity of U_0 enters it.

(1) Clearly

$$\frac{\partial f_i}{\partial \xi_{jk}} = \beta\eta_j\left[\frac{\sum_{u\neq j}\xi_{ju} - \xi_{ji}}{\left(\sum_{u\neq j}\xi_{ju}\right)^2}\right] \quad \text{if} \quad j \neq i = k.$$

Thus

$$\frac{\partial f_i}{\partial \xi_{jk}}(U_0(t), U_0(t-\tau)) = \frac{\beta(n-2)\gamma(t-\tau)}{(n-1)^2\delta(t)}$$

which is independent of $i, j,$ and k just so long as $j\neq i=k$.

(2) Clearly
$$\frac{\partial f_i}{\partial \eta_j} = \frac{\beta\xi_{ji}}{\sum_{k\neq j}\xi_{jk}}(1-\delta_{ij}).$$

Thus

$$\frac{\partial f_i}{\partial \eta_j}(U_0(t), U_0(t-\tau)) = \frac{\beta}{n-1}(1-\delta_{ij}),$$

which is constant.

Computing all partial derivatives in this way and substituting them into (‡) yields after a rearrangement of terms the following system of equations.

$$\dot{h}_i = -\alpha h_i + \frac{\beta\gamma(t-\tau)}{(n-1)^2\delta(t)}\left[(n-1)H_i - H + H^{(i)}\right] + \frac{\beta}{n-1} h^{(i)}(t-\tau), \tag{4.26}$$

where $H_i = \Sigma_{j\neq i} h_{ji}$, $H = \Sigma_{\substack{i,j \\ i\neq j}} h_{ij}$, $H^{(i)} = \Sigma_{j\neq i} h_{ij}$, and $h^{(i)} = \Sigma_{j\neq i} h_j$, $i = 1, 2, \ldots, n$. Moreover,

$$\dot{h}_{jk} = -u h_{jk} + \beta\gamma(t) h_j(t-\tau) + \beta\gamma(t-\tau) h_k, \quad j \neq k \tag{4.27}$$

(II) <u>Equations for the Sums</u> $h = \Sigma_{k=1}^{n} h_k$ <u>and</u> $H = \Sigma_{\substack{j,k \\ j\neq k}} h_{jk}$. In the variational system (‡), the n functions h_i and the $n(n-1)$ functions h_{jk} play different roles, as the equations (4.26) and (4.27) clearly show. This fact is a consequence of the different roles played by x_i and z_{jk} in (✱) itself. We now show that this difference is clearly reflected in the behavior of the sums $h = \Sigma_{k=1}^{n} h_k$ and $H = \Sigma_{\substack{j,k \\ j\neq k}} h_{jk}$ of the two kinds of functions..

It is readily seen that the sum h obeys the simple equation

$$\dot{h} = -\alpha h + \beta h(t-\tau), \tag{4.28}$$

since summing over (4.26) gives

$$\begin{aligned} \dot{h} &= \Sigma_{k=1}^{n} \dot{h}_k \\ &= -\alpha h + \frac{\beta\gamma(t-\tau)}{(n-1)^2\delta(t)}\left[(n-1)\Sigma_{i=1}^{n} H_i - nH + \Sigma_{i=1}^{n} H^{(i)}\right] \\ &\quad + \frac{\beta}{n-1}\Sigma_{i=1}^{n} h^{(i)}(t-\tau), \end{aligned}$$

where $(n-1)\Sigma_{i=1}^{n} H_i - nH + \Sigma_{i=1}^{n} H^{(i)} = 0$ and $\Sigma_i h^{(i)}(t-T) = (n-1)h$. (4.28) is the same equation that is obeyed by the sum $x = \Sigma_{k=1}^{n} x_k$. This is no surprise since (‡) arises by linearizing (✱), and equation (4.28) is already linear. Since h has initial data that satisfies $K_T(h) \neq 0$, by hypothesis, (4.28) is a special case of (4.5). Hence by Proposition 4.1,

$$\lim_{t\to\infty} \frac{h(t-\tau)}{h(t)} = e^{-\tau s(\tau)} \tag{4.36}$$

and

$$\lim_{t\to\infty} \frac{d}{dt}\frac{h(t-\tau)}{h(t)} = 0 . \tag{4.37}$$

H also obeys a simple equation, and one which depends only on γ and h , both of which obey equation (4.28). This equation is readily seen to be

$$\dot{H} = -uH + \beta(n-1)\left[\gamma(t)h(t-\tau) + \gamma(t-\tau)h\right] \tag{4.29}$$

by summing over (4.27). H is therefore independent of the distribution of the functions h_i and h_{jk}.

(III) <u>Uncoupling the Functions</u> h_i <u>From the Functions</u> h_{jk}.

Each of the n equations of (4.26) depends on all n^2 functions h_i and h_{jk}. Because the sums h and H obey equations which are independent of the distribution of the values of these functions, we can nonetheless transform the i^{th} equation into a coupled pair of equations in the single new unknown

$$g_i = \frac{h_i}{\Sigma_{k=1}^{n} h_k} - \frac{1}{n} .$$

It is clear from (4.26) that this transformation can only be accomplished if the function

$$G_i \equiv (n-1)H_i - H + H^{(i)} \quad ,$$

which depends on all functions h_{jk}, $j \neq k$, has some very special properties. That this is true is suggested in (II) by the fact that the summand H is independent of the distribution of the h_{jk}'s. Indeed if we compute $\dot{G}_i$ in terms of (4.27) and rearrange terms, then we find after several cancellations of terms

$$\dot{G}_i = -uG_i + \beta n(n-2)\gamma(t-T)hg_i, \tag{4.31}$$

Since h obeys (4.28), which is independent of the distribution of the unknowns h_i and h_{jk}, G_i's dependence on all the h_{jk}'s is replaced by a dependence on the single unknown g_i.

(4.26) can be written in terms of G_i as

$$\dot{h}_i = -\alpha h_i + \frac{\beta\gamma(t-T)}{(n-1)^2\delta(t)} \quad G_i + \frac{\beta}{n-1} h^{(i)}(t-T) .$$

This equation depends on all h_j, $j=1,2,\ldots,n$ and on g_i. We now transform it into an equation which depends only on g_i. The first step in performing this transformation is to derive an equation for $\lambda_i = \frac{h_i}{h}$.

$$\dot{\lambda}_i = \frac{1}{h}\left(\dot{h}_i - h_i \frac{\dot{h}}{h}\right)$$

$$= \frac{1}{h}\left[-\alpha h_i + \frac{\beta\gamma(t-\tau)G_i}{(n-1)^2\delta} + \frac{\beta}{n-1} h^{(i)}(t-\tau) - h_i\left(-\alpha + \frac{\beta h(t-\tau)}{h}\right)\right]$$

$$= -\beta \frac{h(t-\tau)}{h}\lambda_i + \frac{\beta\gamma(t-\tau)G_i}{(n-1)^2 h\delta} + \frac{\beta}{n-1}\,\frac{h(t-\tau)}{h}\,\lambda^{(i)}(t-\tau),$$

where $\lambda^{(i)} = \frac{h^{(i)}}{h}$. Thus

$$\dot{\lambda}_i = -\frac{\beta h(t-\tau)}{h}\left(\lambda_i - \frac{1-\lambda_i(t-\tau)}{n-1}\right) + \frac{\beta\gamma(t-\tau)G_i}{(n-1)^2 h\delta}.$$

From this equation, we readily complete our transformation using the facts

$$\lambda_i - \frac{1-\lambda_i(t-\tau)}{n-1} = g_i + \frac{1}{n-1} g_i(t-\tau)$$

and $\dot{\lambda}_i = \dot{g}_i$. Thus

$$\dot{g}_i = -\beta\frac{h(t-\tau)}{h}\left(g_i + \frac{1}{n-1} g_i(t-\tau)\right) + \frac{\beta\gamma(t-\tau)G_i}{(n-1)^2 h\delta}. \tag{4.30}$$

(4.30) and (4.31) together completely determine g_i once the initial data of h, γ, and δ is known.

(IV) A Second Order Equation for $g_i = \frac{h_i}{\Sigma_{k=1}^{n} h_k} - \frac{1}{n}$. In order to eliminate G_i from (4.30), we differentiate (4.30) and use (4.31) to eliminate terms involving $\dot{G}_i$. Then we use (4.30) to get rid of G_i itself. To simplify notation, we let $K(t) = \frac{\beta\, h(t-T)}{h}$, $L(t) = \frac{\beta}{(n-1)^2}\,\frac{\gamma(t-T)}{h\delta}$, and $M(t) = \beta n(n-2)\gamma(t-\tau)h$. Then (4.30) and (4.31) can be written as

$$\dot{g}_i = -K\left(g_i + \frac{1}{n-1} g_i(t-\tau)\right) + LG_i \tag{4.30'}$$

and

$$\dot{G}_i = -uG_i + Mg_i. \tag{4.31'}$$

Differentiating (4. 30') gives

$$\ddot{g}_i = -\dot{K}\left(g_i + \frac{1}{n-1} g_i(t-\tau)\right) - K\left(\dot{g}_i + \frac{1}{n-1}\dot{g}_i(t-\tau)\right) + \dot{L}G_i + L\dot{G}_i .$$

Substituting (4. 31') into this equation and rearranging terms gives

$$\begin{aligned} \ddot{g}_i = &\ (LM-\dot{K})g_i - \frac{1}{n-1}\dot{K} g_i(t-\tau) \\ &- K\left(\dot{g}_i + \frac{1}{n-1}\dot{g}_i(t-\tau)\right) \\ &+ (\dot{L}-uL)G_i . \end{aligned} \tag{4. 38}$$

By (4. 30'), $G_i = \frac{K}{L}\left(g_i + \frac{1}{n-1} g_i(t-T)\right) + \frac{\dot{g}_i}{L}$. Substituting this expression into (4. 38) and rearranging terms gives

$$\ddot{g}_i + A(t)\dot{g}_i + B(t)\dot{g}_i(t-T) + C(t)g_i + D(t)g_i(t-T) = 0, \tag{4. 32}$$

where

$$\begin{aligned} A(t) &= K(t) + u - \frac{\dot{L}(t)}{L(t)} \\ &= \beta\left(\frac{h(t-T)}{h}\right) + u + \frac{d}{dt}\log\frac{h\delta}{\gamma(t-T)} , \\ B(t) &= \frac{1}{n-1}K(t) = \frac{\beta}{n-1}\,\frac{h(t-T)}{h} , \\ C(t) &= \dot{K}(t) - L(t)M(t) + K(t)\left(u - \frac{\dot{L}(t)}{L(t)}\right) \\ &= \beta\left[\left(\frac{h(t-T)}{h}\right)^{\bullet} - \frac{\beta n(n-2)}{(n-1)^2}\,\frac{\gamma^2(t-T)}{\delta} + \frac{h(t-T)}{h}\left(u + \frac{d}{dt}\log\frac{h\delta}{\gamma(t-T)}\right)\right], \end{aligned}$$

and

$$\begin{aligned} D(t) &= \frac{1}{n-1}\left(\dot{K}(t) + K(t)\left(u - \frac{\dot{L}(t)}{L(t)}\right)\right) \\ &= \frac{\beta}{n-1}\left[\left(\frac{h(t-T)}{h}\right)^{\bullet} + \left(\frac{h(t-T)}{h}\right)\left(u + \frac{d}{dt}\log\frac{h\delta}{\gamma(t-T)}\right)\right]. \end{aligned}$$

Equation (4.32) would provide little insight into the behavior of $g_i(t)$ as $t \to \infty$ if the coefficients $A(t), B(t), C(t)$, and $D(t)$ were arbitrary continuous functions. Fortunately, these coefficients are constructed from algebraic combinations of the functions $\frac{h(t-T)}{h(t)}$, $\frac{d}{dt}\frac{h(t-T)}{h(t)}$, $\frac{\gamma^2(t-T)}{\delta(t)}$, and $\frac{d}{dt}\frac{h(t)\delta(t)}{\gamma(t-T)}$, where $K_T(h) \neq 0$ and $K_T(\gamma) \neq 0$. We can therefore invoke Proposition 4.1 by letting $\xi_1 = \gamma$, $\xi_2 = h$, and $\eta_1 = \delta$. By Proposition 4.1, the limits $A = \lim_{t\to\infty} A(t)$, $B = \lim_{t\to\infty} B(t)$, $C = \lim_{t\to\infty} C(t)$, and $D = \lim_{t\to\infty} D(t)$ exist. To evaluate these limits, let $k(T) = \beta e^{-Ts(T)}$ and $\theta = \frac{1}{n-1}$ for notational simplicity. Then

$$A = k(T) + u + (-u + \sigma(T))$$

$$= k(T) + \sigma(T),$$

$$B = \theta k(T),$$

$$C = \beta\left[0 - \frac{n(n-2)}{(n-1)^2}\,\sigma(T) e^{-Ts(T)} + e^{-Ts(T)}(u - u + \sigma(T))\right]$$

$$= k(T)\sigma(T)\left(1 - \frac{n(n-2)}{(n-1)^2}\right)$$

$$= \theta^2 k(T)\sigma(T),$$

and

$$D = \frac{\beta\sigma(T)}{n-1}\, e^{-Ts(T)}$$

$$= \theta k(T)\sigma(T).$$

Because the limits A, B, C, and D exist, it is natural to try to compare the behavior of g_i for large t with the behavior of the solution w_i of the following equation with <u>constant</u> coefficients.

$$\ddot{w}_i + A\dot{w}_i + B\dot{w}_i(t-\tau) + Cw_i + Dw_i(t-\tau) = 0. \tag{4.33}$$

For technical reasons, we shall compare an equation related to (4. 32) with an equation related to (4. 33). Thus let $\xi_i = g_i e^{\lambda t}$ and $\eta_i = {}_i e^{\lambda t}$ where λ is any positive constant. To derive an equation for ξ_i, multiply (4. 32) by $e^{\lambda t}$ and use the equalities

$$\ddot{g}_i e^{\lambda t} = \ddot{\xi}_i - 2\lambda \dot{\xi}_i + \lambda^2 \xi_i$$

and

$$\dot{g}_i e^{\lambda t} = \dot{\xi}_i - \lambda \xi_i .$$

We find

$$\ddot{\xi}_i + \bar{A}(t)\dot{\xi}_i + \bar{B}(t)\dot{\xi}_i(t-\tau) + \bar{C}(t)\xi_i + \bar{D}(t)\xi_i(t-\tau) = 0, \tag{4. 34}$$

where

$$\bar{A}(t) = A(t) - 2\lambda ,$$

$$\bar{B}(t) = B(t) ,$$

$$\bar{C}(t) = C(t) + \lambda^2 - \lambda A(t) ,$$

and

$$\bar{D}(t) = D(t) - \lambda B(t) .$$

Similarly, η_i obeys the equation

$$\ddot{\eta}_i + \bar{A}\eta_i + \bar{B}\eta_i(t-T) + \bar{C}\eta_i + \bar{D}\eta_i(t-T) = 0 , \tag{4. 35}$$

where

$$\bar{A} = A - 2\lambda$$
$$= k + \sigma - 2\lambda ,$$

$$\bar{B} = B$$
$$= \theta k ,$$

$$\bar{C} = C + \lambda^2 - \lambda A$$
$$= \theta^2 k\sigma + \lambda^2 - \lambda(k+\sigma),$$

and
$$\bar{D} = D - \lambda B$$
$$= \theta k(\sigma - \lambda).$$

If we can show that ξ_i is a bounded function, say $|\xi_i| \leq k$, then $|g_i| \leq ke^{-\lambda t}$ and since $\lambda > 0$,

$$g_i \equiv \frac{h_i}{\Sigma_{k=1}^{n} h_k} - \frac{1}{n} = o(e^{-\lambda t}), \tag{4.39}$$

which is the first claim of the theorem. To do this, we wish to compare the behavior of ξ_i for large t with that of η_i. If we can do this rigorously and if η_i is a bounded function, our proof of (4.36) will be complete. We now show that this comparison can be carried out and that the boundedness of η_i can be guaranteed by showing that there exists at least one positive λ such that all zeros of the characteristic exponential polynomial

$$G^{(\lambda)}_{\beta n\tau}(s) \equiv s^2 + \bar{A}s + (\bar{B}s + \bar{D})e^{-\tau s} + \bar{C} \tag{4.40}$$

of (4.35) have negative real parts.

The theorems which we shall need to accomplish these aims are applicable to (4.34) when it is written in the matrix form

$$\dot{Z}_i + (V_0 + V_0(t))Z_i + (V_1 + V_1(t))Z_i(t-\tau) = 0, \tag{4.41}$$

where $Z_i = \begin{pmatrix} \xi_i \\ \dot{\xi}_i \end{pmatrix}$, $V_0 = \begin{pmatrix} 0 & -1 \\ \bar{C} & \bar{A} \end{pmatrix}$, $V_1 = \begin{pmatrix} 0 & 0 \\ \bar{D} & \bar{B} \end{pmatrix}$,

$V_0(t) = \begin{pmatrix} 0 & 0 \\ \bar{C}(t) - \bar{C} & \bar{A}(t) - \bar{A} \end{pmatrix}$, and $V_1(t) = \begin{pmatrix} 0 & 0 \\ \bar{D}(t) - \bar{D} & \bar{B}(t) - \bar{B} \end{pmatrix}$.

(4.35) has the matrix form

$$\dot{W}_i + V_o W_i + V_1 W_i(t-T) = 0 \ , \qquad (4.42)$$

where $W_i = \begin{pmatrix} \eta_i \\ \dot{\eta}_i \end{pmatrix}$. The first theorem which we shall need is the following ([3], p. 312).

A sufficient condition in order that all continuous solutions of (4.41) be bounded as $t \to \infty$ is that all solutions of (4.42) be bounded as $t \to \infty$, and that

$$\int_{t_o}^{\infty} \|V_i(t)\| dt < \infty \ , \quad i = 0, 1 \ ,$$

for some $t_o > 0$. The integrals $\int_{t_o}^{\infty} \|V_i(t)\| dt$ are certainly finite for sufficiently large t_o by the inequalities (4.10), (4.11), (4.12), and (4.13) of Proposition 4.1. It therefore remains only to show that all solutions of (4.42) are bounded. We shall be able to show more than this. In fact, by [3], p. 190, all solutions of (4.42) with sufficiently smooth initial data converge to zero as $t \to \infty$ iff all zeros of $G^{(\lambda)}_{\beta n T}(s)$ have negative real parts. We now show that all zeros of $G^{(\lambda)}_{\beta n T}(s)$ have negative real parts for a suitable choice of β, n, T, and λ.

(V) <u>The Zeros of</u> $G^{(\lambda)}_{\beta n T}(s)$. We will show that all zeros of $G^{(\lambda)}_{\beta n T}(s)$ have negative real parts if $k(T) + \sigma(T) > \frac{1}{n-1} k(T)(1 + T\sigma(T))$ and $\lambda > o$ is chosen sufficiently small. This fact relies on the following lemma.

<u>LEMMA 4.2</u>. Suppose that the coefficients of the exponential polynomial

$$G^{(\lambda)}_{\beta n \tau}(s) = s^2 + \bar{A}s + (\bar{B}s + \bar{D})e^{-\tau s} + \bar{C}$$

are positive and $\bar{A} \geq \bar{B} + T\bar{D}$. Then all zeros of $G^{(\lambda)}_{\beta n T}(s)$ have negative real parts.

<u>PROOF.</u> The proof is patterned after [18], in which the closely related exponential polynomial

$$az^2 + bz + \beta z e^{-z} + c$$

is studied. Letting $z=\tau s$, $\tau>o$, the equation $G^{(\lambda)}_{\beta n\tau}(s)=0$ becomes

$$f(z) \equiv z^2 + Ez + (Fz+H)e^{-z} + J = 0,$$

where $E=\bar{A}\tau$, $F=\bar{B}\tau$, $H=\bar{D}\tau^2$, and $J=\bar{C}\tau^2$. The zeros of $f(z)$ are the same as the zeros of $G^{(\lambda)}_{\beta n\tau}(s)$ for all $\tau>0$. For $\tau=0$, it is obvious that all zeros of $G^{(\lambda)}_{\beta n\tau}(s)=0$ have negative real parts if λ is chosen sufficiently small, by the positivity of $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$. In the following, we therefore consider the zeros of $f(z)$ for $\tau>0$. In this case, E, F, H, and J are all positive if λ is sufficiently small.

The main fact used in our analysis is Cauchy's Index Theorem: Suppose $w=f(z)$ is an analytic function of z in a simply connected domain D bounded by a closed curve C, where $f(z)\neq 0$ for $z\in C$. If z traverses C in a counterclockwise direction, then $f(z)$ will traverse a closed curve in the w-plane and the number of zeros of $f(z)$ in D is equal to the number of times the w-contour encircles the origin.

This basic theorem is used to study the zeros of $w=f(z)$ in the following way. As z traverses C in a counterclockwise direction, w may cross the real axis. Let δ_+ be the number of times that w crosses the real axis in a counterclockwise direction relative to the origin (i. e., from quadrant IV to quadrant I or from quadrant II to quadrant III), and let δ_- be the number of times w crosses the real axis in a clockwise direction relative to the origin. The number of zeros of $f(z)$ in D then equals $\frac{1}{2}(\delta_+ - \delta_-)$.

We apply Cauchy's Index Theorem to the semicircular domain

$$D: \mathrm{Re}(z)>0 \quad \text{and} \quad |z|<R$$

in the z-plane. If $|z|\geq R$ and $\mathrm{Re}(z)\geq o$, where R is chosen sufficiently large, then

$$|z^2| > |Ez + (Fz+H)e^{-z} + J|$$

and $z^2 \neq 0$. Rouche's theorem therefore implies that $f(z) \neq 0$ for $|z| \geq R$ and $Re(z) \geq 0$. D is fixed once and for all by making a choice of a sufficiently large R. For this choice of D, all the zeros of $f(z)$ in the right half plane will lie in D.

To apply Cauchy's theorem to this domain D, we divide its boundary curve C into two parts

$$A: \quad Re(z) = 0 \quad \text{and} \quad |z| \leq R$$

and

$$B: \quad Re(z) > 0 \quad \text{and} \quad |z| = R.$$

Consider A. Letting $z=iy$ gives

$$g(iy) = -y^2 + H\cos y + Fy\sin y + J + i(Ey + Fy\cos y - H\sin y),$$

where $J > 0$ and $E \geq F+H$ by hypothesis. If $y=0$, then $f(0)=H+J>0$. If $0 < y \leq R$, then

$$\begin{aligned} Im(g(iy)) &= Ey + Fy\cos y - H\sin y \\ &= y\left(E + F\cos y - H\frac{\sin y}{y}\right) \\ &\geqslant 0, \end{aligned}$$

since $\left|\frac{\sin y}{y}\right| \leq 1$. Thus w is in either quadrant I or quadrant II. $f(iR)$ is in quadrant II since R is large and $f(0)$ is in quadrant I. If $-R \leq y < 0$, then $Im(g(iy)) = y(E + F\cos y - H\frac{\sin y}{y}) \leq 0$ and w is in either quadrant III or quadrant IV. Thus as z traverses A from $+iR$ to $-iR$, w crosses the real axis once in a clockwise direction relative to the origin.

Now consider $f(z)$ as z traverses B from $-iR$ to $+iR$. Since $f(z)$ is dominated by $-z^2$, the net number of times that $f(z)$ crosses the real axis in a counterclockwise direction relative to the origin is once. We have therefore shown that $\frac{1}{2}(\delta_+ - \delta_-)=0$, and thus that $f(z)$ has no zeros in D, or

for that matter in the right half plane. This completes the proof of the Lemma.

We remark that [18] goes on to give necessary and sufficient conditions for his equation to have zeros only in the left half plane even when the inequalities analogous to $J>0$ and $E \geq F+H$ are not satisfied. These conditions seem to be difficult to apply in the present case.

We now apply Lemma 4.2 to the present example. When $\tau=0$, the zeros of $G^{(\lambda)}_{\beta n\tau}(s)$ obviously have negative real parts for sufficiently small $\lambda>0$ by the positivity of A, B, C, and D. Consider the case $\tau>0$, where λ is chosen so small that $\bar{A}, \bar{B}, \bar{C}$, and $\bar{D}$ are all positive. The condition $J>0$ is satisfied since $\bar{C}>0$ and $\tau>0$. The condition $E \geq F+H$ becomes $\bar{A}\tau \geq \bar{B}\tau+\bar{D}\tau^2$ or $\bar{A} \geq \bar{B}+\tau\bar{D}$. Since $\bar{A}=k(\tau)+\sigma(\tau)-2\lambda$, $\bar{B}=\theta k(\tau)$, and $\bar{D}=\theta k(\tau)(\sigma(\tau)-\lambda)$, this inequality becomes

$$k(\tau)+\sigma(\tau)-2\lambda \geq \frac{1}{n-1} k(\tau)\left[1+\tau(\sigma(\tau)-\lambda)\right].$$

If, as hypothesized, $k(\tau)+\sigma(\tau)>\frac{1}{n-1} k(\tau)(1+\tau\sigma(\tau))$, we can certainly find a sufficiently small positive λ for which $E \geq F+H$. This shows that all the zeros of $G^{(\lambda)}_{\beta n\tau}(s)$ have negative real parts if $k(\tau)+\sigma(\tau)>\frac{1}{n-1} k(\tau)(1+\tau\sigma(\tau))$ and λ is chosen sufficiently small. For $\lambda=\omega_1$ chosen in this way, we can therefore conclude that

$$\frac{h_i(t)}{\sum_{k=1}^{n} h_k(t)} - \frac{1}{n} = o\left(e^{-\omega_1 t}\right) \tag{4.24}$$

This completes the first part of the theorem by proving (4.24). We now turn to the proof of (4.25), using the result (4.24).

(VI) <u>An Equation for</u> $g_{jk}=\frac{h_{jk}}{\Sigma_{m\neq j}h_{jm}}-\frac{1}{n-1}$. In this step, we prove the existence of an $\omega_2>0$ for which

$$\frac{h_{jk}(t)}{\sum_{\substack{m=1\\ m\neq j}}^{n} h_{jm}(t)} - \frac{1}{n-1} = o\left(e^{-\omega_2 t}\right) \tag{4.43}$$

for all $j \neq k$. (4.43) is the analog of the statement proved in (V) that there exists an $\omega_1 > 0$ such that

$$\frac{h_i(t)}{\sum_{k=1}^{n} h_k(t)} - \frac{1}{n} = o\left(e^{-\omega_1 t}\right), \quad i=1,2,\ldots,n. \tag{4.24}$$

(4.24) was proved by deriving an equation for $g_i = \frac{h_i}{\Sigma_{k=1}^{n} h_k} - \frac{1}{n}$ whose solutions behave like $o(e^{-\omega_1 t})$ for large t. (4.43) is proved by deriving an equation for $g_{jk} = \frac{h_{jk}}{\Sigma_{m\neq j} h_{jm}} - \frac{1}{n-1}$ whose solution converges exponentially to zero as $t \to \infty$. This equation will have the form

$$\dot{g}_{jk} = \Lambda_j \left(G_{jk} - g_{jk} \right) \tag{4.44}$$

where Λ_j can be written as

$$\Lambda_j = \frac{d}{dt} \log \Gamma_j . \tag{4.45}$$

Integrating (4.44) gives

$$g_{jk}(t) = e^{-\int_0^t \Lambda_j du}\left[g_{jk}(0) + \int_0^t e^{\int_0^v \Lambda_j du} \Lambda_j G_{jk} dv\right].$$

Since $\exp[-\int_w^t \Lambda_j du] = \frac{\Gamma_j(w)}{\Gamma_j(t)}$ and $\Lambda_j = \dot{\Gamma}_j / \Gamma_j$,

$$g_{jk}(t) = \frac{g_{jk}(0)\Gamma_j(0)}{\Gamma_j(t)} + \int_0^t \frac{\dot{\Gamma}_j(v) G_{jk}(v)}{\Gamma_j(t)} dv. \tag{4.46}$$

The facts proved in (I)-(V) will be shown to imply that $G_{jk}(t) = o(e^{-\omega_1 t})$ for some $\omega_1 > 0$, and that

$$\dot{\Gamma}_j(t) = e^{\sigma(\tau)t}\left(\mu + e^{-\kappa t} M_j(t)\right), \qquad (4.47)$$

where $\mu \neq 0$, $\kappa > 0$, and $M_j(t)$ is bounded. From these properties the following conclusions can be drawn.

Since

$$\Gamma_j(t) = \Gamma_j(0) + e^{\sigma(\tau)t}\left[\frac{\mu}{\sigma(\tau)}\left(1 - e^{-\sigma(\tau)t}\right) + e^{-\kappa t} R_j(t)\right],$$

where $R_j(t)$ is a bounded function, the term

$$\frac{g_{jk}(0)\,\Gamma_j(0)}{\Gamma_j(t)}$$

in (4.46) converges exponentially to zero as $t \to \infty$. Consider the term

$$\int_0^t \frac{\dot{\Gamma}_j(v)\, G_{jk}(v)}{\Gamma_j(t)}\, dv \qquad (4.48)$$

in (4.46). Since

$$\frac{\dot{\Gamma}_j(v)}{\Gamma_j(t)} = e^{-\sigma(\tau)(t-v)}\left[\frac{\mu + e^{-\kappa v} M_j(v)}{\Gamma_j(0)e^{-\sigma(\tau)t} + \frac{\mu}{\sigma(\tau)}\left(1 - e^{-\sigma(\tau)t}\right) + e^{-\kappa t} R_j(t)}\right],$$

given any $\epsilon > 0$ we can find a $T(\epsilon)$ such that $t \geq v \geq T(\epsilon)$ implies

$$\left|\frac{\dot{\Gamma}_j(v)}{\Gamma_j(t)}\right| \leq e^{-\sigma(\tau)(t-v)}\left[\frac{\mu + \epsilon}{\frac{\mu}{\sigma(\tau)} - \epsilon}\right]. \qquad (4.49)$$

Let $e = \frac{\mu}{2\sigma(T)}$ and let T_0 be the corresponding $T(e)$. Then the integral (4.48) can be broken into the two parts

$$\frac{1}{\Gamma_j(t)}\int_0^{T_0}\dot{\Gamma}_j(v)G_{jk}(v)\,dv \quad \text{and} \quad \int_{T_0}^{t}\frac{\dot{\Gamma}_j(v)}{\Gamma_j(t)}G_{jk}(v)\,dv,$$

for $t \geq T_0$. The first part converges exponentially to zero as $t \to \infty$ because $(\Gamma_j(t))^{-1}$ does. The second part converges exponentially to zero by (4.49) because $G_{jk}(t)$ does.

We now derive equation (4.44) for g_{jk}. Writing g_{jk} as

$$g_{jk} = H_{jk} - \frac{1}{n-1},$$

where $H_{jk} = \frac{h_{jk}}{H^{(j)}}$ and $H^{(j)} = \Sigma_{m \neq j} h_{jm}$, we first derive an equation for $H^{(j)}$ and then one for H_{jk}. It is readily seen by summing over $k \neq j$ in (4.27) that $H^{(j)}$ obeys the equation

$$\dot{H}^{(j)} = -uH^{(j)} + \beta(n-1)\gamma h_j(t-\tau) + \beta\gamma(t-\tau)h^{(j)}, \tag{4.50}$$

where $h^{(j)} = \Sigma_{k \neq j} h_k$. (4.50) along with (4.27) suffice to derive the following equation for H_{jk}.

$$\begin{aligned}\dot{H}_{jk} &= \frac{1}{H^{(j)}}\left[\dot{h}_{jk} - h_{jk}\frac{\dot{H}^{(j)}}{H^{(j)}}\right] \\ &= \frac{1}{H^{(j)}}\Big[-uh_{jk} + \beta\gamma h_j(t-\tau) + \beta\gamma(t-\tau)h_k \\ &\qquad - h_{jk}\Big(-u + \frac{\beta(n-1)\gamma h_j(t-\tau) + \beta\gamma(t-\tau)h^{(j)}}{H^{(j)}}\Big)\Big],\end{aligned}$$

and

$$\dot{H}_{jk} = \frac{\beta}{H^{(j)}}\Big[\gamma h_j(t-\tau) + \gamma(t-\tau)h_k - H_{jk}\Big((n-1)\gamma h_j(t-\tau) + \gamma(t-\tau)h^{(j)}\Big)\Big]. \tag{4.51}$$

(4.51) is now used to derive equation (4.44) for $g_{jk} = H_{jk} - \frac{1}{n-1}$ by rearranging terms in (4.51). Thus

$$\dot{g}_{jk} = \dot{H}_{jk}$$

$$= \frac{\beta}{H^{(j)}} \Big\{ \gamma \ell_j(t-\tau) + \gamma(t-\tau)\ell_k - \Big(H_{jk} - \frac{1}{n-1}\Big)\Big[(n-1)\gamma \ell_j(t-\tau) + \gamma(t-\tau)\ell^{(j)}\Big]$$

$$- \Big[\gamma \ell_j(t-\tau) + \frac{1}{n-1}\gamma(t-\tau)\ell^{(j)}\Big]\Big\}$$

$$= \frac{\beta}{H^{(j)}} \Big\{ \gamma(t-\tau)\Big[\ell_k - \frac{1}{n-1}\ell^{(j)}\Big] - g_{jk}\Big[(n-1)\gamma \ell_j(t-\tau) + \gamma(t-\tau)\ell^{(j)}\Big]\Big\}.$$

Letting

$$\Lambda_j = \frac{\beta\left[(n-1)\gamma \ell_j(t-\tau) + \gamma(t-\tau)\ell^{(j)}\right]}{H^{(j)}}$$

and

$$G_{jk} = \frac{\gamma(t-\tau)\left[\ell_k - \frac{1}{n-1}\ell^{(j)}\right]}{(n-1)\gamma \ell_j(t-\tau) + \gamma(t-\tau)\ell^{(j)}}$$

we find the equation

$$\dot{g}_{jk} = \Lambda_j\left(G_{jk} - g_{jk}\right),$$

which has the same form as (4.44).

To complete the proof, we must show that (1) G_{jk} converges exponentially to zero as $t \to \infty$, and that (2) Λ_j can be written as $\Lambda_j = \frac{d}{dt}\log \Gamma_j$ where Γ_j can be written in the form given in (4.59)

(1) Consider G_{jk}. We must show that G_{jk} converges exponentially to zero as $t \to \infty$. Dividing numerator and denominator of G_{jk} by $h = \Sigma_{k=1}^{n} h_k$ and invoking the definition $g_i \equiv \frac{h_i}{h} - \frac{1}{n}$, we find that

$$G_{jk} = \frac{\gamma(t-\tau)\left(g_k - \frac{1}{n-1}\Sigma_{m\neq j}\, g_m\right)}{\frac{(n-1)\gamma h_j(t-\tau)}{h} + \frac{\gamma(t-\tau)\,h^{(j)}}{h}}\,.$$

The denominator can be rewritten as follows.

$$\frac{(n-1)\gamma h_j(t-\tau)}{h} + \frac{\gamma(t-\tau)h^{(j)}}{h} = (n-1)\gamma\left(\frac{h(t-\tau)}{h}\right)\left(\frac{h_j(t-\tau)}{h(t-\tau)}\right) + \gamma(t-\tau)\left(\frac{h^{(j)}}{h}\right)$$

$$= \gamma(t-\tau)\left[(n-1)\left(\frac{\gamma}{\gamma(t-\tau)}\right)\left(\frac{h(t-\tau)}{h}\right)\left(g_j(t-\tau)+\frac{1}{n}\right) + \Sigma_{m\neq j}\, g_m + \frac{n-1}{n}\right].$$

Thus

$$G_{jk} = \frac{g_k - \frac{1}{n-1}\Sigma_{m\neq j}\, g_m}{(n-1)\left(\frac{\gamma}{\gamma(t-\tau)}\right)\left(\frac{h(t-\tau)}{h}\right)\left(g_j(t-\tau)+\frac{1}{n}\right) + \Sigma_{m\neq j}\, g_m + \frac{n-1}{n}}\,. \tag{4.52}$$

By Proposition 4.1,

$$\lim_{t\to\infty}\left(\frac{\gamma}{\gamma(t-\tau)}\right)\left(\frac{h(t-\tau)}{h}\right) = \lim_{t\to\infty}\frac{\gamma}{\gamma(t-\tau)}\lim_{t\to\infty}\frac{h(t-\tau)}{h}$$

$$= e^{\tau s(\tau)}\, e^{-\tau s(\tau)}$$

$$= 1,$$

whereas by (4.24), there exists a $\omega_1>0$ such that $g_k=o(e^{-\omega_1 t})$, for all $k=1,2,\ldots,n$. By (4.52), we readily conclude that $G_{jk}=o(e^{-\omega_1 t})$ as well, $j\neq k$.

(2) Consider

$$\Lambda_j(t) = \frac{\beta\left[(n-1)\gamma h_j(t-\tau) + \gamma(t-\tau)\, h^{(j)}\right]}{H^{(j)}}.$$

We first show that Λ_j can be written as $\Lambda_j = \frac{d}{dt}\log \Gamma_j$. Integrating (4.50) we find

$$H^{(j)}(t) = e^{-ut}\left[H^{(j)}(0) + \beta\int_0^t e^{uv}\left[(n-1)\gamma h_j(v-\tau) + \gamma(v-\tau)\, h^{(j)}\right]dv\right].$$

Thus

$$\Lambda_j(t) = \frac{\beta e^{ut}\left[(n-1)\gamma h_j(t-\tau) + \gamma(t-\tau)\, h^{(j)}\right]}{H^{(j)}(0) + \beta\int_0^t e^{uv}\left[(n-1)\gamma h_j(v-\tau) + \gamma(v-\tau)\, h^{(j)}\right]dv}.$$

Letting

$$\Gamma_j(t) \equiv H^{(j)}(0) + \beta\int_0^t e^{uv}\left[(n-1)\gamma h_j(v-\tau) + \gamma(v-\tau)\, h^{(j)}\right]dv$$

we therefore find that $\Lambda_j = \frac{d}{dt}\log \Gamma_j$.

It remains only to show that Γ_j can be written as in (4.47).

$$\begin{aligned}
\dot{\Gamma}_j(t) &= \beta e^{ut}\left[(n-1)\gamma h_j(t-\tau) + \gamma(t-\tau)\, h^{(j)}\right] \\
&= \beta e^{ut}\left[(n-1)\gamma h(t-\tau)\left(\frac{h_j(t-\tau)}{h(t-\tau)}\right) + \gamma(t-\tau)\, h\left(\frac{h^{(j)}}{h}\right)\right] \\
&= \beta e^{ut}\left[(n-1)\gamma h(t-\tau)\left(g_j(t-\tau) + \frac{1}{n}\right) + \gamma(t-\tau)\, h\left(\Sigma_{k\neq j}\, g_k + \frac{n-1}{n}\right)\right].
\end{aligned}$$

Since $K_T(\gamma)\neq 0$ and $K_T(h)\neq 0$, we can by (4.20) write γ and h as

$$\gamma = e^{s(T)t}(c_\gamma + e^{-k_\gamma t}M_\gamma(t))$$

and

$$h = e^{s(T)t}(c_h + e^{-k_h t}M_h(t)),$$

where $c_\gamma = 0$, $c_h = 0$, $k_\gamma > 0$, $k_h > 0$, and $M_\gamma(t)$ and $M_h(t)$ are bounded. Thus

$$\dot{\Gamma}_j(t) = \beta e^{-\tau s(\tau)} e^{\sigma(\tau)t}\Big[(n-1)(c_\gamma + e^{-k_\gamma t}M_\gamma(t))\cdot (c_h + e^{-k_h(t-\tau)}M_h(t-\tau))(g_j(t-\tau)+\tfrac{1}{n}) + (c_\gamma + e^{-k_\gamma(t-\tau)}M_\gamma(t-\tau))(c_h + e^{-k_h t}M_h(t))\cdot (\Sigma_{k\neq j}\, g_k + \tfrac{n-1}{n})\Big].$$

Since $g_k = o(e^{-\omega_1 t})$, $k=1,2,\ldots,n$,

$$\dot{\Gamma}_j(t) = e^{\sigma(\tau)t}\Big[\frac{2(n-1)}{n}k(\tau)c_\gamma c_h + e^{-kt}M(t)\Big],$$

where k is a number such that $0<k<\min[\frac{\omega_1}{2}, k_\gamma, k_h]$ and $M(t)$ is bounded. This completes the proof of Theorem 4.1.

4. THE GRADATION OF STABILITY PROPERTIES WITH RESPECT TO n AND T.

By Theorem 4.1, if the coefficients are chosen to satisfy $\beta > 0$, $\sigma(T)>0$, and $k(T)+\sigma(T)>\frac{1}{n-1}k(T)(1+T\sigma(T))$, then the ratios of the solutions become uniformly distributed as $t\to\infty$. These conditions imply the following relations between systems with different n and T but the same α, β, and u.

A) For every $T\geq 0$ such that $\sigma(T)>0$, there exists an $n=n(T)$ such that the theorem holds for n and T. This is true because

$$\lim_{n\to\infty}\frac{1}{n-1}k(T)(1+T\sigma(T))=0.$$

B) If Theorem 4.1 is true for fixed $n=n_o$ and $T=T_o$, then it is true for all $n\geq n_o$ and $T=T_o$, since then

$$\begin{aligned} k(T_o)+\sigma(T_o) &> \frac{1}{n_o-1}\,k(T_o)(1+T_o\sigma(T_o)) \\ &\geq \frac{1}{n-1}\,k(T_o)\,(1+T_o\sigma(T_o)). \end{aligned}$$

Thus the stability properties of the variational system are <u>graded</u> in n.

C) As in (B), it readily follows that if Theorem 4.1 holds for $n=n_o$ and $T=T_o$, then it holds for $n=n_o$ and all T in a neighborhood of T_o. The case where $\alpha>\beta>o$ is of special interest since then the outputs $x_i(t)$ of (✱) approach zero as $t\to\infty$ for all $n\geq 3$ and $T\geq 0$. This is the case in which (✱) has a prediction theoretic interpretation.

<u>COROLLARY 4.1.</u> If $\alpha>\beta>0$, then $\sigma(T)$ and $k(T)$ are monotone increasing functions of T, $T\geq 0$, such that $\sigma(0)=\sigma\equiv u+2(\beta-\alpha)$ and $k(0)=\beta$.

<u>PROOF.</u> By Lemma 4.1, for any fixed $T\geq 0$, the zero $s(T)$ of largest real part of $P_T(s)=s+\alpha-\beta e^{-Ts}$ is real. The proof of Lemma 4.1 also

shows that $s(\tau)<0$ whenever $\alpha>\beta>0$. Thus

$$-|s(T)|+\alpha=\beta e^{T|s(T)|} \tag{4.53}$$

Suppose any two nonnegative values T_1 and T_2 of T are given such that

$$|s(T_2)|>|s(T_1)|\geq 0.$$

Then by (4.53)

$$\beta(e^{T_1|s(T_1)|}-e^{T_2|s(T_2)|})=|s(T_2)|-|s(T_1)|>0.$$

Since $\beta>0$,

$$e^{T_1|s(T_1)|}>e^{T_2|s(T_2)|}.$$

and thus

$$T_1|s(T_1)|>T_2|s(T_2)|\geq 0. \tag{4.54}$$

In particular, $T_1>0$. Since $T_1>0$ and $|s(T_2)|>|s(T_1)|$

$$T_1|s(T_2)|>T_1|s(T_1)| \tag{4,55}$$

(4.55) along with (4.54) implies

$$T_1|s(T_2)|>T_2|s(T_2)|,$$

or

$$T_1>T_2.$$

We have hereby shown that $T_1\leq T_2$ implies $|s(T_2)|\leq|s(T_1)|$. Since $(T)=u+2s(T)=u-2|s(T)|$, $T_1\leq T_2$ implies $\sigma(T_1)\leq\sigma(T_2)$. $\sigma(T)$ is therefore a monotone increasing function of T, for $T\geq 0$.

$s(0)$ satisfies the equation

$$P_0(s) = s + \alpha - \beta = 0.$$

Thus $s(0) = \beta - \alpha$ and

$$\begin{aligned}\sigma(0) &= u - 2|s(0)| = u - 2|\beta - \alpha| \\ &= u - 2(\alpha - \beta) \\ &= u + 2(\beta - \alpha) \\ &= \sigma.\end{aligned}$$

By (4.53), $k(T) = \alpha - |s(T)|$ where $|s(T)|$ is monotone decreasing in $T \geq 0$. Thus $k(T)$ is monotone increasing in $T \geq 0$. Also $k(0) = \beta e^{0|s(0)|} = \beta$. This completes the proof.

By Corollary 4.1, if $\sigma > 0$ and $T=0$, then

$$k(0) + \sigma(0) = \beta + \sigma \geq \frac{\beta}{n-1} = \frac{k(0)}{n-1}(1 + 0\sigma(0))$$

so that Theorem 4.1 holds for all $n \geq 3$ when $\sigma > 0$ and $T=0$. Since $k(T)$ and $\sigma(T)$ are monotone increasing functions of $T \geq 0$, if $\sigma > 0$ then there exists an interval $[0, \omega(n))$, where $\omega(n)$ is monotone increasing in n and $\lim_{n \to \infty} \omega(n) = \infty$, such that Theorem 4.1 holds for all $T \in [0, \omega(n))$.

CHAPTER V

GLOBAL RATIO LIMIT THEOREMS FOR GENERAL COMPLETE GRAPHS WITH LOOPS

1. GENERAL COMPLETE GRAPHS WITH LOOPS

In Theorem 3.2 we considered a special case of an input-free system whose coefficient matrix is

$$P = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & & \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}.$$

The general input-free system with this coefficient matrix P is

$$\dot{x}_i(t) = -\alpha x_i(t) + \beta \Sigma_{k=1}^{n} x_k(t-\tau) y_{ki}(t), \qquad i=1,2,\ldots,n \qquad (5.1)$$

$$y_{jk}(t) = \frac{z_{jk}(t)}{\Sigma_{m=1}^{n} z_{jm}(t)} \qquad j,k=1,2,\ldots,n \qquad (5.2)\ (*)$$

and

$$\dot{z}_{jk}(t) = -uz_{jk}(t) + \beta x_j(t-\tau) x_k(t), \qquad j,k=1,2,\ldots,n, \qquad (5.3)$$

and the coefficient graph is (e.g. when n=3) given in Figure 23.

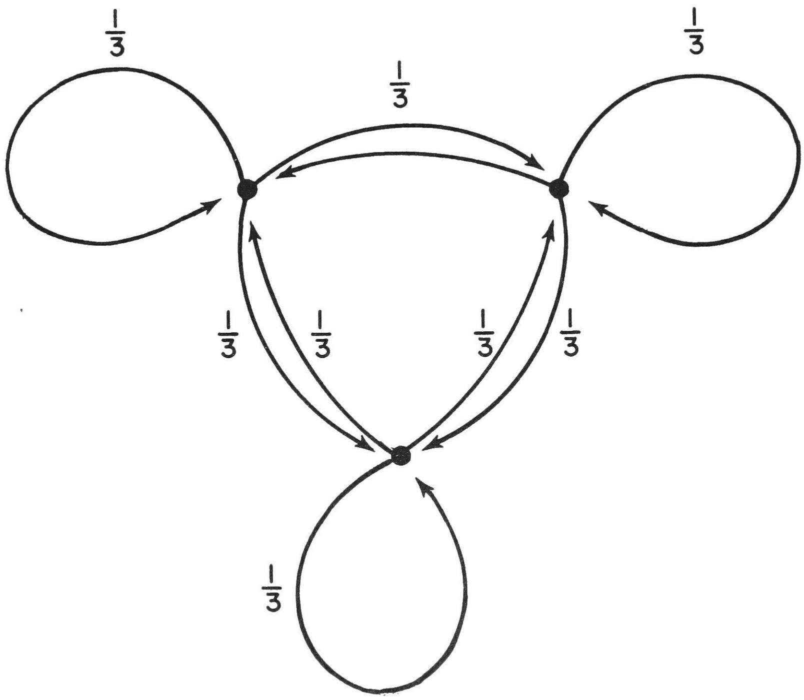


Fig. 23

The system of Theorem 3. 2 is characterized by the choices $n = 2$ and $\tau = 0$. In this case, we observed that the ratios X_i and y_{jk} of a system with $\beta > 0$, $\sigma > 0$, and positive initial data have limits Q_i and P_{jk} which always obey the constraints $Q_i = P_{ji}$, $i, j = 1, 2$, and that any probability distribution Q_1, Q_2 can arise as a limit.

In particular, when $X_i(0) = y_{ji}(0)$, for all $i, j = 1, 2$, $X_i(t) = y_{ji}(t) =$ constant so that $Q_i = P_{ji}$ and the system does not "forget" its initial data. This fact differs substantially from the result of Theorem 3. 1 for the 3-graph without loops which says that $Q_i = \frac{1}{3}$ and $P_{jk} = \frac{1}{2}(1-\delta_{jk})$ if $\sigma > 0$, or that the initial data are forgotten.

In this chapter, we generalize this fact to systems with arbitrary $n \geq 2$ and $\tau \geq 0$. We then show how this fact is reflected in the eigenvalues of an equation related to the variational system of (*) and contrast these eigenvalues with those of Chapter 4. The first theorem describes the limiting behavior of the ratios y_{jk} and $X_i = \dfrac{x_i}{\sum_{k=1}^{n} x_k}$ as $t \to \infty$ for any $n \geq 2$ and $\tau \geq 0$.

THEOREM 5. 1. Suppose $\beta > 0$ and $\sigma(\tau) > 0$, and let (*) have arbitrary nonnegative and continuous initial data. Then the limits $Q_i = \lim_{t\to\infty} X_i(t)$ and $P_{jk} = \lim_{t\to\infty} y_{jk}(t)$ exist and satisfy the equations $P_{ji} = Q_i$. Moreover $Q_i \in [m_i, M_i]$ where $m_i = \min\{X_i(0), y_{ki}(0): k = 1, 2, \ldots, n\}$ and $M_i = \max\{X_i(0), y_{ki}(0): k=1, 2, \ldots, n\}$.

We consider only the case in which all initial data is positive. All other cases are then easily treated using the convention that $\frac{0}{0} = 0$ for the ratios X_i and y_{jk}. The proof depends on the following lemma.

LEMMA 5. 1. The ratios X_i and y_{jk} obey the following equations

$$\dot{X}_i = A\Sigma_{k=1}^{n} X_k(t-\tau)(y_{ki} - X_i) \tag{5.4}$$

and

$$\dot{y}_{jk} = B_j(X_k - y_{jk}), \tag{5.5}$$

with $A = \frac{\beta x(t-T)}{x}$ and $B_j = \frac{\beta x_j(t-T)x}{z(j)}$, where $x = \Sigma_{k=1}^{n} x_k$ is a positive solution of

$$\dot{x} = -\alpha x + \beta x(t-T) \tag{5.6}$$

and $z^{(j)} = \Sigma_{m=1}^{n} z_{jm}$.

PROOF. (5.4) is proved as follows.

$$\dot{X}_i = \frac{1}{x}\left(\dot{x}_i - x_i \frac{\dot{x}}{x}\right)$$

$$= \frac{1}{x}\left[-\alpha x_i + \beta \sum_{k=1}^{n} x_k(t-\tau) y_{ki} - x_i\left(-\alpha + \frac{\beta x(t-\tau)}{x}\right)\right]$$

$$= \frac{\beta x(t-\tau)}{x}\left(\sum_{k=1}^{n} X_k(t-\tau) y_{ki} - X_i\right)$$

$$= A \sum_{k=1}^{n} X_k(t-\tau)\left(y_{ki} - X_i\right).$$

(5.5) is proved in a similar way, namely

$$\dot{y}_{jk} = \frac{1}{z^{(j)}}\left(\dot{z}_{jk} - z_{jk} \frac{\dot{z}^{(j)}}{z^{(j)}}\right)$$

$$= \frac{1}{z^{(j)}}\left[-u z_{jk} + \beta x_j(t-\tau) x_k - z_{jk}\left(-u + \frac{B x_k(t-\tau) x}{z^{(k)}}\right)\right]$$

$$= B_j\left(X_k - y_{jk}\right).$$

The following estimates on the coefficients A and B_j shall also be needed.

LEMMA 5.2. $\lim_{t\to\infty} A(t) = k(T) > 0$, and there exists a T such that $t \geq T$

implies $B_j \geq \theta X_j(t-T)$ where $\theta > 0$.

PROOF. That $\lim_{t\to\infty} A(t)=k(T)$ is obvious by Proposition 4.1 and the positivity of all initial data since then $K_T(x)\neq 0$.

By (✱) and Proposition 4.1,

$$\dot{z}^{(j)} = -uz^{(j)} + \beta x_j(t-\tau)x$$

$$= -uz^{(j)} + \beta X_j(t-\tau)x(t-\tau)x.$$

Since $x(t)$ obeys (5.6) and is positive, we have by Proposition 4.1 that

$$x(t) = e^{s(\tau)t}\left(c_1 + e^{-\eta t}M(t)\right),$$

where $c_1 > 0$, $\eta > 0$, and $M(t)$ is nonnegative and bounded. Thus

$$\dot{z}^{(j)} = -uz^{(j)} + \beta X_j(t-\tau)e^{2s(\tau)t}\cdot\left(c_1^2 e^{-\tau s(\tau)} + e^{-\eta t}N(t)\right), \tag{5.7}$$

where $N(t)$ is nonnegative and bounded. Integrating (5.7), we find after letting $N = \sup_{t\in[0,\infty)}\left(c_1^2 e^{-Ts(T)} + e^{-\eta t}N(t)\right)$ that

$$z^{(j)}(t) = e^{-ut}\left(z^{(j)}(0) + \beta\int_0^t e^{\sigma(\tau)v}X_j(v-\tau)\cdot\left(c_1^2 e^{-\tau s(\tau)} + e^{-\eta v}N(v)\right)dv\right)$$

$$\leq e^{-ut}\left(z^{(j)}(0) + \beta N\int_0^t e^{\sigma(\tau)v}\,dv\right)$$

$$= e^{-ut}\left(z^{(j)}(0) + \frac{\beta N}{\sigma(\tau)}\left(e^{\sigma(\tau)t} - 1\right)\right).$$

Since $\sigma(T)>0$, there exists a T such that $t\geq T$ implies

$$z^{(j)}(t)\leq \frac{2\beta N}{\sigma(T)} e^{(\sigma(T)-u)t}$$

Thus

$$B_j = \frac{\beta x_j(t-T)x}{z^{(j)}}$$

$$\geq \frac{\sigma(T)}{2N} e^{(u-\sigma(T))t} x_j(t-T)x$$

$$= \frac{\sigma(T)}{2N} e^{(u-\sigma(T))t} X_j(t-T)x(t-T)x$$

$$= \frac{\sigma(T)}{2N} X_j(t-T)(c_1^2 e^{-Ts(T)} + e^{-\eta t} N(t))$$

$$\geq \frac{c_1^2 e^{-Ts(T)} \sigma(T)}{2N} \quad X_j(t-T)$$

Letting $\theta = \dfrac{c_1^2 e^{-Ts(T)}\sigma(T)}{2N}$ completes the proof.

The remainder of the proof of Theorem 5.1 requires that we introduce the functions $Y_i = \max\{y_{ki}: k=1,2,\ldots,n\}$ and $y_i = \min\{y_{ki}: k=1,2,\ldots,n\}$. The following facts concerning these functions are obvious from Lemma 5.1 and the positivity of A, B_j, and X_j, $j=1,2,\ldots,n$.

1) If for any $t_o \geq 0$ $X_i(t_o)\in[y_i(t_o), Y_i(t_o)]$, then $X_i(t)\in[y_i(t), Y_i(t)]$ for all $t\geq t_o$, where $y_i(t)$ is monotone increasing and $Y_i(t)$ is monotone decreasing for all $t\geq t_o$.

2) If $X_i(0) > Y_i(0)$, then $X_i(t)$ is monotone decreasing and all $y_{ki}(t)$ are monotone increasing until the first time $t=t_1>0$ at which $X_i(t)=Y_i(t)$. Thereafter $Y_i(t)$ is monotone decreasing and $y_i(t)$ is monotone increasing by (1),

so that $Y_i(t)$ changes sign at most once and $y_i(t)$ is always monotone increasing.

3) If $X_i(0) < y_i(0)$, then $X_i(t)$ is monotone increasing and all $y_{ki}(t)$ are monotone decreasing until the first time $t = t_1 > 0$ at which $X_i(t) = y_i(t)$. Thereafter $y_i(t)$ is monotone increasing by (1), so that $\dot{y}_i(t)$ changes sign at most once, and $Y_i(t)$ is always monotone decreasing. These alternatives are illustrated in Figure 24.

Cases (1), (2), and (3) exhaust all possibilities. We conclude that $X_i(t)$ and all $y_{ki}(t)$, $k=1,2,\ldots,n$, lie in $[m_i, M_i]$ for all $t \geq 0$, and that $\dot{Y}_i$ or $\dot{y}_i$ change sign at most once. In particular the limits $y_i(\infty) = \lim_{t\to\infty} y_i(t)$ and $Y_i(\infty) = \lim_{t\to\infty} Y_i(t)$ exist since y_i and Y_i are bounded. The remainder of the proof falls into three cases that exhaust all possibilities.

<u>Case A.</u> $X_i > Y_i$ <u>for all</u> $t \geq 0$. Then by (2), X_i is monotone decreasing and all y_{ki} are monotone increasing. Hence all limits Q_i and P_{ki} exist and $Q_i \geq P_{ki}$. It is readily shown using (5.4) that $\ddot{X}_i$ is bounded. Thus by Lemma 2.3, $\lim_{t\to\infty} \dot{X}_i(t) = 0$. By (5.4) and Lemma 5.2,

$$0 = k(T) \lim_{t\to\infty} \Sigma_{k=1}^{n} X_k(t-T)(P_{ki} - Q_i)$$

or $\lim_{t\to\infty} \Sigma_{k=1}^{n} X_k(t)(P_{ki} - Q_i) = 0$. Since $X_k \geq 0$ and $P_{ki} - Q_i \leq 0$ for all $k=1,2,\ldots,n$, $(P_{ki} - Q_i) \lim_{t\to\infty} X_k(t) = 0$ for all $k=1,2,\ldots,n$. Either $P_{ki} = Q_i$ or $\lim_{t\to\infty} X_k(t) = 0$ for all $k=1,2,\ldots,n$. Since $X_k(t) \geq m_k > 0$, $P_{ki} = Q_i$ for all $k=1,2,\ldots,n$.

<u>Case B.</u> $X_i < y_i$ <u>for all</u> $t \geq 0$. The proof is the same as for Case A.

<u>Case C.</u> $X_i(t) \in [y_i(t), Y_i(t)]$ <u>for all</u> $t \geq t_0$. If $y_i(\infty) = Y_i(\infty)$ we are done since then all Q_i and P_{ki} exist and equal $y_i(\infty)$. The only remaining case is $Y_i(\infty) - y_i(\infty) \equiv \epsilon_i > 0$. We now show that this case cannot arise.

Consider $y_i(t)$. For convenience we write $y_i(t)$ as $y_{k(t),\, i}(t)$ to explicitly display the index $k = k(t)$ of that $y_{ki}(t)$ which equals $y_i(t)$ at every

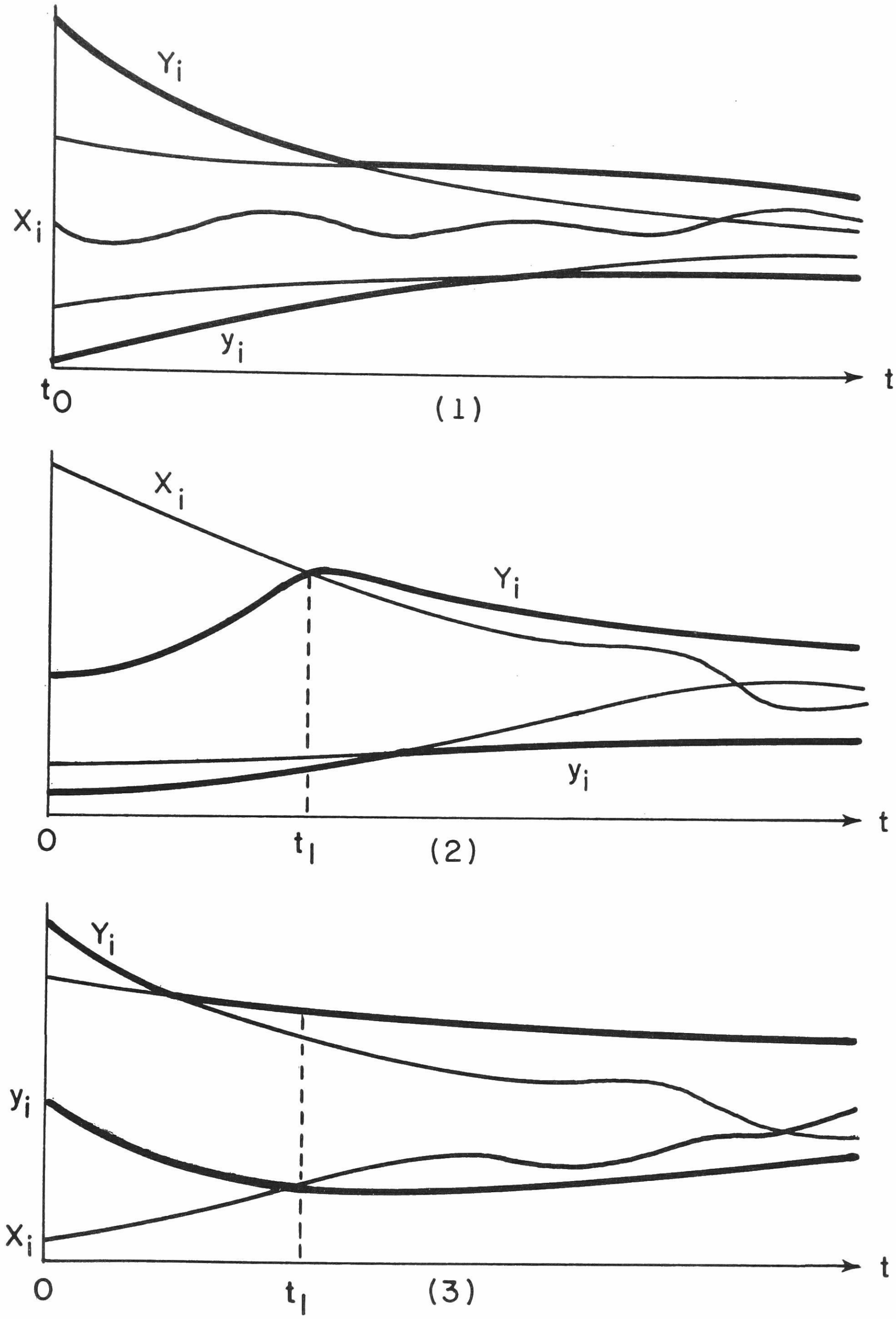


Fig. 24

t. Since each $\dot{y}_{ki}(t)$ is bounded, the integer-valued function $k(t)$ is constant in intervals of positive length. Using also the fact that $\ddot{y}_{ki}$ is bounded for every $k=1,2,\ldots,n$, Lemma 2.3 and (5.5) imply $\lim_{t\to\infty} B_{k(t)}(t)|X_i(t)-y_{k(t),i}(t)| = 0$. By Lemma 5.2,

$$B_{k(t)}(t) \geq \frac{\sigma(T)}{M} X_{k(t)}(t-T)$$

$$\geq \frac{\sigma(T)}{M} m_{k(t)}$$

$$\geq \frac{\sigma(T)}{M} \min\{m_i : i=1,2,\ldots,n\} > 0$$

for t sufficiently large. Thus $\lim_{t\to\infty}(X_i(t)-y_{k(t),i}(t))=0$. Similarly, $\lim_{t\to\infty}(X_i(t)-y_{k(t),i}(t))=0$, so that $\lim_{t\to\infty}(Y_{k(t),i}(t)-y_{k(t),i}(t))=0$, or $\epsilon_i=0$, which completes the proof.

2. GRADATION OF STABILITY PROPERTIES WITH RESPECT TO THE LAG TIME T.

In Corollary 4.1, we showed that $\alpha > \beta > 0$ implies $\sigma(T)$ is a monotone increasing function of $T \geq 0$ and $\sigma(0) = \sigma \equiv u + 2(\beta - \alpha)$. Thus if $\beta > 0$ and $\sigma(T_o) > 0$, then Theorem 5.1 holds for all systems with $T \geq T_o$ and any $n \geq 2$. Moreover if $\sigma > 0$, then Theorem 5.1 holds for all systems with $T \geq 0$ and $n \geq 2$. That is, the condition needed to guarantee convergence of the ratios as $t \to \infty$ becomes <u>weaker</u> as the lag time becomes <u>larger</u>. We therefore say that the stability properties of (✱) are <u>graded</u> in the lag time T.

This gradation of stability properties in T can be heuristically interpreted if we think of (✱) as a flow on a graph in the obvious way. Let each edge e_{ij} of the graph associated with (✱) have a length, which we take to be 1 for all edges. Then the lag time T can be interpreted as the inverse

velocity $\frac{1}{v}$ of the flows along all the edges. Theorem 5.1 says that if the ratios have limits when the flow velocity is V_0, then they have limits also for all smaller flow velocities. If we regard the velocity of the flow as an indicator of the strength of the interaction between vertices, then Corollary 4.1 says that it gets harder to guarantee the stability of this flow as the interaction gets stronger. This fact is intuitively plausible.

The fact that we can guarantee stability for all flow velocities if $u > 2(\alpha - \beta) > 0$ has the following interpretation. The parameters α, β, and u can be thought of as characterizing the materials which go into the construction of each separate vertex and each separate edge of (✱). From this point of view, the parameters α, β, and u are "local" quantities, since they do not take into consideration the various ways in which the vertices and edges can interact. In constructing these vertices and edges, it is natural to ask the following question: Can we choose our materials once and for all in such a way that (✱) will be stable no matter how strongly the vertices and edges interact? , Theorem 5.1 and Corollary 4.1 guarantee that the answer to this question is "yes" because $\sigma(0) = \sigma$.

3. THE VARIATIONAL SYSTEM.

The familiar notion of a variational system was introduced in Chapter 4 to discuss the linearized behavior of a general complete graph without loops. The variational system which we studied in Chapter 4 had the form

$$\dot{w}(t) = f_\xi(u_0(t), u_0(t-\tau))\, w(t) + f_\eta(u_0(t), u_0(t-\tau))\, w(t-\tau), \quad (\ddagger)$$

where U_0 is any positive uniform solution of the general complete graph without loops. To facilitate comparison with the results of Chapter 4, in this chapter we again study the variational system with U_0 chosen as an arbitrary positive uniform solution of the general complete graph <u>with</u> loops.

For this system, a positive uniform solution is one whose initial data satisfies (1) $x_i(\vartheta)=\gamma(\vartheta)>0$, $\vartheta\in[-T,0]$ for all i, and (2) $z_{jk}(0)=\delta(0)>0$ for all j and k.

Since the strategy for studying the variational system of the graph with loops is essentially the same as the strategy employed to study the graph without loops, we shall often state only the most important points in the following discussion. Before actually proving anything about the variational system of (✱), we qualitatively summarize the main differences between the variational system of a graph with loops and one without loops. The solution of the variational system of the graph without loops is denoted, as in Chapter 4, by $W=(h_1,h_2,\ldots,h_n,h_{12},\ldots,h_{n,n-1})$. The solution of the variational system of the graph with loops is denoted by

$$W=(h_1,h_2,\ldots,h_n,h_{11},h_{12},\ldots,h_{n,n-1},h_{nn})$$

to facilitate comparison between the two systems. No confusion shall arise in this way.

Firstly we bring together some salient points concerning the variational system without loops. In Chapter 4, we found an equation of the form

$$\ddot{\xi}_i+\bar{A}(t)\dot{\xi}_i+\bar{B}(t)\dot{\xi}_i(t-\tau)+\bar{C}(t)\xi_i+\bar{D}(t)\xi_i(t-\tau)=0 \tag{5.8}$$

for the unknown

$$\xi_i=\left(\frac{h_i}{\sum_{k=1}^{n}h_k}-\frac{1}{n}\right)e^{\lambda t}$$

where λ is a sufficiently small positive constant. Since the limits $\bar{A}=\lim_{t\to\infty}\bar{A}(t)$, $\bar{B}=\lim_{t\to\infty}\bar{B}(t)$, $\bar{C}=\lim_{t\to\infty}\bar{C}(t)$, and $\bar{D}=\lim_{t\to\infty}\bar{D}(t)$ exist, we compared this equation with

$$\ddot{\eta}_i + \bar{A}\dot{\eta}_i + \bar{B}\dot{\eta}_i(t-\tau) + \bar{C}\eta_i + \bar{D}\eta_i(t-\tau) = 0, \tag{5.9}$$

and showed that all the zeros of the characteristic exponential polynomial

$$G(s) = s^2 + \bar{A}s + (\bar{B}s + \bar{D})e^{-\tau s} + \bar{C} \tag{5.10}$$

of (5. 9) have negative real parts for an appropriate choice of λ, β, n, and T. From this follows that

$$\frac{h_i(t)}{\sum_{k=1}^{n} h_k(t)} - \frac{1}{n} = o(e^{-\lambda t}).$$

In this chapter, we shall find an equation of the form

$$\ddot{g}_i + A(t)\dot{g}_i + C(t)g_i = 0 \tag{5.11}$$

for the unknown

$$g_i = \frac{h_i}{\sum_{k=1}^{n} h_k} - \frac{1}{n}$$

of the variational system with loops. Notice that no terms of the form $B(t)\dot{g}_i(t-T)$ or $D(t)g_i(t-T)$ occur in (5. 11) as they did in (5. 6). This dramatically simplifies the mathematics of a graph with loops. Moreover, the limits $A = \lim_{t\to\infty} A(t)$ and $C = \lim_{t\to\infty} C(t)$ exist. A is positive, if $\sigma(T)$ is, and C is <u>zero</u>. Thus we compare the solutions of (5. 11) for large times with those of the simple equation

$$\ddot{w}_i + A\dot{w}_i = 0. \tag{5.12}$$

The characteristic polynomial of (5.12) is

$$\hat{G}(s)=s(s+A) \tag{5.13}$$

which has roots $s=0, A$ where A is negative iff $\sigma(T)+k(T)>0$. Thus not all zeros of $\hat{G}(s)$ have negative real parts, as was the case for $G(s)$ in (5.10). It is not difficult to see that the zero solution of (5.13) corresponds to the fact that the limits Q_i and P_{jk} of (✱) are not unique.

With this introduction in mind, we now prove the following theorem concerning the variational system of a general complete graph with loops. We shall need the functions $\sigma(T)$ and $K_T(f)$ to state this theorem, and refer the reader to Chapter 4 for their definitions. We again concern ourselves with the behavior of the ratios $H_i = \frac{h_i}{\Sigma_{k=1}^n h_k}$ and $H_{jk} = \frac{h_{jk}}{\Sigma_{m=1}^n h_{jm}}$ as $t \to \infty$.

THEOREM 5.2. Let $n \geq 2$ and $T \geq 0$ be chosen arbitrarily. Suppose for this choice that $\beta > 0$ and $\sigma(T) > 0$. Consider the variational system (‡) of (✱), where $U_o(t)$ is a fixed but arbitrary positive uniform solution of (✱). For arbitrary initial data of (‡) satisfying $K_T(\Sigma_{k=1}^n h_k) \neq 0$, the limits

$$\tilde{Q}_i = \lim_{t \to \infty} \frac{h_i(t)}{\Sigma_{k=1}^n h_k(t)}$$

and

$$\tilde{P}_{jk} = \lim_{t \to \infty} \frac{h_{jk}(t)}{\Sigma_{m=1}^n h_{jm}(t)}$$

exist and satisfy the equations

$$\tilde{P}_{jk} = \frac{\tilde{Q}_j + \tilde{Q}_k}{1 + n\tilde{Q}_j} . \tag{5.14}$$

(5.14) shows that linearizing (✱) changes the distribution of its solutions as $t \to \infty$. (5.14) replaces the condition $P_{jk} = Q_k$ which appeared in Theorem 5.1. These two conditions are compatible for positive $\tilde{Q}_j$ iff $\tilde{P}_{jk} = \tilde{Q}_k = \frac{1}{n}$, since if

$$\tilde{Q}_k = \tilde{P}_{jk} = \frac{\tilde{Q}_j + \tilde{Q}_k}{1 + n\tilde{Q}_j}$$

then
$$\tilde{Q}_k + n\tilde{Q}_j\tilde{Q}_k = \tilde{Q}_j + \tilde{Q}_k$$

and
$$\tilde{Q}_k = \frac{1}{n}.$$

The strategy for proving Theorem 5.1 is the same as that used to prove Theorem 4.1. We therefore display only the most important equations.

<u>PROOF.</u> (I) <u>The Variational System in Component Form.</u> For any positive uniform solution $\mathcal{U}_0 = (\gamma, \gamma, \ldots, \gamma, \delta, \delta, \ldots, \delta)$, the variational system (ǂ) can be written out in component form as

$$\dot{h}_i = -\alpha h_i + \frac{\beta}{n} h(t-\tau) + \frac{\beta\gamma(t-\tau)}{n^2 \delta(t)}\left[(n-1)H^{(i)} - \sum_{k \neq i} H^{(k)}\right] \tag{5.15}$$

and

$$\dot{h}_{jk} = -u h_{jk} + \beta\left[\gamma(t-\tau) h_k + \gamma h_j(t-\tau)\right], \tag{5.16}$$

where $H^{(j)} = \Sigma_{k=1}^{n} h_{kj}$.

Letting $h = \Sigma_{k=1}^{n} h_k$ and summing (5.15) over all i gives

$$\dot{h} = -\alpha h + \beta h(t-T), \tag{5.17}$$

as in (4.28).

(II) A Second Order Equation for $g_i = \frac{h_i}{\Sigma_{k=1}^{n} h_k} - \frac{1}{n}$. We now show that the variable $g_i = \frac{h_i}{h} - \frac{1}{n}$ satisfies a coupled pair of equations of the form

$$\dot{g}_i = -Dg_i + EG_i \tag{5.18}$$

and

$$\dot{G}_i = -uG_i + Fg_i \tag{5.19}$$

Firstly an equation for $H_i = \frac{h_i}{h}$ is derived. By (5.15) and (5.16)

$$\dot{H}_i = \frac{1}{h}\left(\dot{h}_i - h_i \frac{\dot{h}}{h}\right)$$

$$= \frac{1}{h}\left[-\alpha h_i + \frac{\beta}{n} h(t-\tau) + \frac{\beta\gamma(t-\tau)}{n^2\delta}\left((n-1)H^{(i)} - \Sigma_{k\neq i} H^{(k)}\right)\right.$$
$$\left. - h_i\left(-\alpha + \frac{\beta h(t-\tau)}{h}\right)\right]$$

$$= -\frac{\beta h(t-\tau)}{h}\left(H_i - \frac{1}{n}\right)$$
$$+ \frac{\beta\gamma(t-\tau)}{n^2\delta h}\left[(n-1)H^{(i)} - \Sigma_{k\neq i} H^{(k)}\right].$$

Letting $D = \frac{\beta h(t-T)}{h}$, $E = \frac{\beta\gamma(t-T)}{n^2\delta h}$, and $G_i = (n-1)H^{(i)} - \Sigma_{k\neq i} H^{(k)}$, we find

$$\dot{H}_i = -D(H_i - \frac{1}{n}) + EG_i$$

Since $g_i = H_i - \frac{1}{n}$ and $\dot{g}_i = \dot{H}_i$,

$$\dot{g}_i = -DH_i + EG_i\ . \tag{5.18}$$

We now derive equation (5.19) for G_i, where $F = \beta n^2 \gamma(t-T)h$.

By (5.14),

$$\dot{G}_i = (n-1)\dot{H}^{(i)} - \Sigma_{k\neq i} \dot{H}^{(k)}$$

$$= -uG_i + \beta(n-1)\left[n\gamma(t-\tau)h_i + \gamma h(t-\tau)\right]$$

$$- \beta\left[n\gamma(t-\tau)\Sigma_{k\neq i} h_k + (n-1)\gamma h(t-\tau)\right]$$

$$= -uG_i + \beta n\gamma(t-\tau)\left[(n-1)h_i - \Sigma_{k\neq i} h_k\right]$$

$$= -uG_i + \beta n\gamma(t-\tau)\left[n h_i - h\right]$$

$$= -uG_i + \beta n^2\gamma(t-\tau) h g_i$$

$$= -uG_i + F g_i \tag{5.19}$$

(5.18) and (5.19) allow us to derive a second order equation for g_i. Differentiating (5.18) gives

$$\ddot{g}_i = -\dot{D}g_i - D\dot{g}_i + \dot{E}G_i + E\dot{G}_i . \tag{5.20}$$

Substituting (5.19) into (5.20) gives

$$\ddot{g}_i = (EF-\dot{D})g_i - D\dot{g}_i + (\dot{E}-uE)G_i .$$

By (5.18)

$$G_i = \frac{\dot{g}_i + Dg_i}{E} .$$

Substituting this into (5.20) gives

$$\ddot{g}_i + A(t)\dot{g}_i + C(t)g_i = 0 \tag{5.21}$$

where

$$A(t) = D(t) + u - \frac{\dot{E}(t)}{E(t)}$$

$$= \frac{\beta h(t-\tau)}{h} + u - \frac{d}{dt}\log\frac{\gamma(t-\tau)}{\delta h}$$

and

$$C(t) = D(t)\left(u - \frac{\dot{E}(t)}{E(t)}\right) + \dot{D}(t) - EF(t)$$

$$= \frac{\beta h(t-\tau)}{h}\left(u - \frac{d}{dt}\log\frac{\gamma(t-\tau)}{\delta h}\right) + \beta\left(\frac{h(t-\tau)}{h}\right)^{\cdot} - \frac{\beta^2\gamma^2(t-\tau)}{\delta} .$$

Because γ and h obey (5.17), where by hypothesis $K_T(\gamma) \neq 0$ and $K_T(h) \neq 0$, Proposition 4.1 implies that the limits $A = \lim_{t\to\infty} A(t)$ and $C = \lim_{t\to\infty}$ exist. These limits equal

$$A = k(\tau) + u - (u - \sigma(\tau))$$

$$= k(\tau) + \sigma(\tau) > 0,$$

and

$$C = k(\tau)\left[u-(u-\sigma(\tau))\right] + 0 - \sigma(\tau)k(\tau)$$

$$= k(\tau)\sigma(\tau) - k(\tau)\sigma(\tau)$$

$$= 0.$$

We therefore compare the behavior of (5.21) for large t with the behavior of the solution of the following system with constant coefficients.

$$\ddot{w}_i + A\dot{w}_i = 0. \tag{5.22}$$

This comparison is made rigorous in [3]. (5.22) implies

$$\dot{w}_i + Aw_i = \lambda \tag{5.23}$$

where λ is an arbitrary constant. (5.23) can be immediately integrated (when $A \neq 0$) to give

$$w_i = e^{-At} w_i(0) + \frac{\lambda(1-e^{-At})}{A}.$$

Since $A>0$, $\lim_{t\to\infty} w_i(t)$ exists and equals $\frac{\lambda}{A}$, which is an arbitrary constant since λ is arbitrary. From this follows the existence of all limits

$$\lim_{t\to\infty} g_i(t) = \lim_{t\to\infty} \frac{h_i(t)}{\sum_{k=1}^{n} h_k(t)} - \frac{1}{n},$$

and thus of all limits

$$\tilde{Q}_i = \lim_{t\to\infty} \frac{h_i(t)}{\sum_{k=1}^n h_k(t)} .$$

It is easily seen from this argument that the limits $\tilde{Q}_i$ exist for all values of A, and hence for all values of α, β, and u. $A \geq 0$ merely guarantees that these limits are finite.

(III) <u>An Equation for</u> $H_{jk} = h_{jk} / \Sigma_{m=1}^n h_{jm}$. We now use the existence of the limits $\tilde{Q}_i$ to prove the existence of the limits $\tilde{P}_{jk} = \lim_{t\to\infty} H_{jk}(t)$. The proof proceeds exactly as in section (VI) of Theorem 4.1. We therefore merely sketch the essentials.

H_{jk} once again obeys an equation of the form

$$\dot{H}_{jk} = \Lambda_j (G_{jk} - H_{jk}) \tag{5.24}$$

where $\Lambda_j = \frac{d}{dt} \log \Gamma_j$. In the present case

$$G_{jk} = \frac{\gamma(t-\tau) h_k + \gamma h_j (t-\tau)}{\gamma(t-\tau) h + n\gamma h_j (t-\tau)} \tag{5.25}$$

and

$$\Gamma_j = H^{(j)}(0) + \beta \int_0^t e^{uv} \left[\gamma(v-\tau) h + n\gamma h_j (v-\tau) \right] dv .$$

It is readily shown that

$$\lim_{t\to\infty} G_{jk}(t) = \frac{\tilde{Q}_k + \tilde{Q}_j}{1 + n\tilde{Q}_j} \tag{5.26}$$

To see this, divide numerator and denominator of (5.25) by $\gamma(t-T)h$. Then

$$G_{jk} = \frac{H_k + \frac{\gamma}{\gamma(t-\tau)} \frac{h_j(t-\tau)}{h}}{1 + \frac{n\gamma}{\gamma(t-\tau)} \frac{h_j(t-\tau)}{h}}$$

$$= \frac{H_k + \frac{\gamma}{\gamma(t-\tau)} \frac{h(t-\tau)}{h} H_j(t-\tau)}{1 + n\frac{\gamma}{\gamma(t-\tau)} \frac{h(t-\tau)}{h} H_j(t-\tau)} . \tag{5.27}$$

By Proposition 4.1,

$$\lim_{t\to\infty} \frac{\gamma}{\gamma(t-T)} \frac{h(t-T)}{h} = 1 ,$$

while by section (II), the limits $\tilde{Q}_i$ exist. Letting $t\to\infty$ in (5.27) therefore immediately gives (5.26).

The remainder of the proof that $\tilde{P}_{jk}$ exists and equals

$$\lim_{t\to\infty} G_{jk}(t) = \frac{\tilde{Q}_k + \tilde{Q}_j}{1+n\tilde{Q}_j}$$

now goes through just as in Theorem 4.1 (VI). The only change is that G_{jk} does not in the present case necessarily converge to zero.

Theorem 5.2 shows that the variational system of (*) also has stability properties which are graded in the time lag T.

CHAPTER VI

CONCLUDING REMARKS

This thesis introduces a prediction theory whose goal is to discuss the prediction of individual events, in a fixed order, and at prescribed times. We have studied herein only the simplest cases of this prediction theory, but these cases already reveal some mathematical features which seem bound to reoccur in later discussions of more complicated cases.

1. THE GEOMETRY OF LEARNING.

A basic fact seems to be that the way in which a system learns from its input experience depends critically on its geometry. Thus an input-free outstar (Figure 7) never forgets, an input-free complete 3-graph without loops (Figure 4) eventually forgets everything (if $\sigma > 0$), and an input-free complete 2-graph with loops (Figure 15) remembers quite well (even if $\sigma > 0$). Chapter IV and V suggest, moreover, that the differences in the way in which these "small" systems of a given type of geometry remember generalize to systems composed of arbitrarily many vertices n interacting with an arbitrary lag time $\tau \geq 0$.

Because of these differences in learning due to differences in geometry, we can ask the following kind of question within our systems without ever contemplating a change of dynamics. Given a set of sequences of events to be predicted, what kind of geometry is best suited for the job?

2. SOME PATHS FOR FUTURE RESEARCH.

Concerning the present theory, it is clear that a great deal of further mathematical work remains to be done on how lists of events ABC... composed of more than two letters and occurring at varying speeds are learned and stored by these machines. A substantial amount of heuristic work has already been done in this direction and will appear in an applied context. Some computer work has also been done and is illustrated in

Appendix B. This question is really one of classification, since we wish to assign to every semistochastic matrix P the function space of vector input functions from which the system characterized by P can benefit from experience. That the geometry induced in the machine by P is closely related to the geometry of this function space is clearly seen by the following almost trivial example.

Consider the outstar of Figure 25 whose border is initially uniform under two different input vector functions:

$$(1)\quad I_1(t) = I_2(t) = \sum_{k=0}^{\infty} J(t - k(w + W)),$$

$$I_j(t) \equiv 0, \quad j \neq 1, 2,$$

and

$$(2)\quad I_2(t) = I_3(t) = \sum_{k=0}^{\infty} J(t - k(w + W)),$$

$$I_j(t) \equiv 0, \quad j = 1, 4, \ldots, n.$$

In case (1), $\lim_{t\to\infty} y_{12}(t) = 1$ by Theorem 2.2. In case (2), $y_{12}(t) = y_{13}(t) = \frac{1}{2}$ for all $t \geq 0$ by symmetry. An outstar whose source stands for the symbol A can learn the list AB but not the list BC.

From this example, it is clear that if we want a system to be able to learn from a large set of vector inputs, then we are well advised to construct a system whose edges connect many different vertices. On the other hand, Theorem 3.1 shows us that there exist systems in which every edge is connected to every other edge which nonetheless forget everything that they are taught very quickly. Theorem 3.2 suggests a way out of this dilemma by suggesting that we connect many different vertices together, but also connect vertices with themselves by way of loops.

The above remarks suggest that the number of connections between different vertices is an important factor in determining how these systems learn. So too is the actual distribution of the semistochastic coefficients P which weigh the connections. For example, consider a complete n

graph without loops that has uniform initial data. Any vector input function of the form $I = (J, J, \ldots, J)$ can occur in this graph without distorting the uniform distribution of the graph's data. A uniformly distributed complete graph does not "see" a uniformly distributed input. Changing the distribution of coefficients P changes the input vectors which the graph can see.

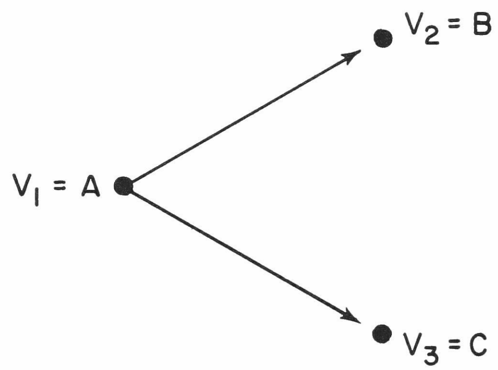


Fig. 25

APPENDIX A

PROOF OF THEOREM 2.4

PART I.

A. Derivation of

$$\frac{D_2(k)}{D_3(k)} - \frac{D_2(k-1)}{D_3(k-1)} = \frac{\Phi_k\left(D_2(k-1)+(n-2)D_3(k-1)\right)}{D_3(k-1)\left(\Psi_k + D_2(k-1)+(n-2)D_3(k-1)\right)}$$

The equation for each w_j in (2.25) is a first-order linear equation in each interval of the form $[n(k-1), n(k))$ since $w_1(t-\tau)$ and I_2 are known functions. For $t \in [n(k-1), n(k))$,

$$\dot{w}_j(t) = -\alpha w_j(t) + \beta\lambda_{j,k-1}\, w_1(t-\tau) + \delta_{j2}\, I_2(t).$$

In particular,

$$w_j(n(k)) = e^{-\alpha n(1)}\Big[w_j(n(k-1)) + \beta\lambda_{jk}\int_{n(k-1)}^{n(k)} e^{\alpha(\xi-n(k-1))}\, w_1(\xi-\tau)\,d\xi + \delta_{j2}\int_{n(k-1)}^{n(k)} e^{\alpha(\xi-n(k-1))}\, I_2(\xi)\,d\xi\Big]. \qquad \text{(A. 1)}$$

Similarly (2.26) has the integral form

$$D_j(k) = D_{1j}(n(k)) = e^{-un(1)}\Big[D_j(k-1) + \beta\int_{n(k-1)}^{n(k)} e^{u(\xi-n(k-1))}\, w_1(\xi-\tau)\, w_j(\xi)\,d\xi\Big]. \qquad \text{(A. 2)}$$

Substituting (A. 2) into (A. 1) and forming the ratio $D_2(k)/(D_2(k)+(n-2)D_3(k))$ gives

$$\frac{D_2(k)}{D_2(k)+(n-2)D_3(k)} = \frac{D_2(k-1)+\lambda_{2,k-1}\Psi_k+\Phi_k}{D_2(k-1)+(n-2)D_3(k-1)+\Psi_k+\Phi_k}, \tag{A. 3}$$

where Φ_k and Ψ_k are defined in (2. 28) and (2. 29). By (A. 3),

$$D_2(k)\left[1-\frac{D_2(k-1)+\lambda_{2,k-1}\Psi_k+\Phi_k}{D_2(k-1)+(n-2)D_3(k-1)+\Psi_k+\Phi_k}\right]$$

$$=(n-2)D_3(k)\left[\frac{D_2(k-1)+\lambda_{2,k-1}\Psi_k+\Phi_k}{D_2(k-1)+(n-2)D_3(k-1)+\Psi_k+\Phi_k}\right],$$

or

$$D_2(k)[(n-2)D_3(k-1)+\Psi_k(1-\lambda_{2,k-1})] = (n-2)D_3(k)[D_2(k-1)+\lambda_{2,k-1}\Psi_k+\Phi_k].$$

Since

$$\lambda_{2,k-1} = D_2(k-1)/(D_2(k-1)+(n-2)D_3(k-1)),$$

$$(n-2)D_2(k)D_3(k-1)\left(1+\frac{\Psi_k}{D_2(k-1)+(n-2)D_3(k-1)}\right) =$$

$$(n-2)D_3(k)\left[D_2(k-1)\left(1+\frac{\Psi_k}{D_2(k-1)+(n-2)D_3(k-1)}\right)+\Phi_k\right]$$

Thus

$$\frac{D_2(k)}{D_3(k)}\left(1+\frac{\Psi_k}{D_2(k-1)+(n-2)D_3(k-1)}\right)$$

$$=\frac{D_2(k-1)}{D_3(k-1)}\left(1+\frac{\Psi_k}{D_2(k-1)+(n-2)D_3(k-1)}+\frac{\Phi_k}{D_3(k-1)}\right),$$

$$\frac{D_2(k)}{D_3(k)} = \frac{D_2(k-1)}{D_3(k-1)} + \frac{\Phi_k / D_3(k-1)}{1 + \Psi_k / (D_2(k-1) + (n-2) D_3(k-1))},$$

and

$$\frac{D_2(k)}{D_3(k)} - \frac{D_2(k-1)}{D_3(k-1)} = \frac{\Phi_k (D_2(k-1) + (n-2) D_3(k-1))}{D_3(k-1)(\Psi_k + D_2(k-1) + (n-2) D_3(k-1))}$$

B. Derivation of $D_2(k) + (n-2)D_3(k) = R^k(V_k + c)$.

By (A. 1) and (A. 2), we know that

$$D_2(k) = R(D_2(k-1) + \lambda_{2,\,k-1}\Psi_k + \Phi_k) \qquad \text{(A. 4)}$$

and

$$D_3(k) = R(D_3(k-1) + \lambda_{3,\,k-1}\Psi_k) \qquad \text{(A. 5)}$$

where $R = e^{-u(w+W)}$. Thus

$$D_2(k) + (n-2)D_3(k) = R[D_2(k-1) + (n-2)D_3(k-1) + (\lambda_{2,\,k-1} + (n-2)\lambda_{3,\,k-1})\Psi_k + \Phi_k]$$

$$= R[D_2(k-1) + (n-2)D_3(k-1) + \Psi_k + \Phi_k] \,.$$

Iterating this identity gives

$$D_2(k) + (n-2)D_3(k) = \sum_{j=1}^{k} R^{k-j+1}(\Psi_j + \Phi_j) + R^k (n-1) D_{12}(0)$$

$$= R^k(V_k + c). \qquad \text{(A. 6)}$$

C. Derivation of

$$\frac{1}{D_3(k)} = \frac{R^{-k}}{D_{12}(0)} \prod_{j=1}^{k} \left(\frac{V_{j-1} + c}{V_{j-1} + c + \Psi_j R^{-(j-1)}} \right).$$

By (A. 5) and (A. 6),

$$D_3(k) = R\left(D_3(k-1) + \lambda_{3,k-1}\Psi_k\right)$$

$$= RD_3(k-1)\left(1 + \frac{\Psi_k}{D_2(k-1) + (n-2)D_3(k-1)}\right)$$

$$= RD_3(k-1)\left(1 + \frac{\Psi_k}{R^{k-1}(V_{k-1} + c)}\right)$$

$$= RD_3(k-1)\frac{R^{k-1}(V_{k-1} + c) + \Psi_k}{R^{k-1}(V_{k-1} + c)}$$

$$= RD_3(k-1)\left(\frac{V_{k-1} + c + \Psi_k R^{-(k-1)}}{V_{k-1} + c}\right).$$

Iterating this identity gives

$$D_3(k) = R^k D_{12}(0) \prod_{j=1}^{k} \left(\frac{V_{j-1} + c + \Psi_j R^{-(j-1)}}{V_{j-1} + c} \right)$$

and

$$\frac{1}{D_3(k)} = \frac{R^{-k}}{D_{12}(0)} \prod_{j=1}^{k} \left(\frac{V_{j-1} + c}{V_{j-1} + c + \Psi_j R^{-(j-1)}} \right).$$

PART II

A. The sequence $\Psi_1, \Psi_2, \ldots$ is monotone increasing.

$w_1(t) \le w_1(t+m(1))$ for all $t \ge 0$ since $I_1(t) \le I_1(t+n(1))$. In particular, $w_1(v-\tau+n(k-1)) \le w_1(v-\tau+n(k))$ for $v \in [0, n(1))$. Thus by (2.29),

$$\Psi_{k+1} = \beta^2 \int_0^{n(1)} e^{(u-\alpha)v} w_1(v-\tau+n(k)) \int_0^v e^{\alpha u} w_1(u-\tau+n(k))\,du\,dv$$

$$\geq \beta^2 \int_0^{n(1)} e^{(u-\alpha)v} w_1(v-\tau+n(k-1)) \int_0^v e^{\alpha u} w_1(u-\tau+n(k-1))\,du\,dv$$

$$= \Psi_k .$$

B. The sequence $\Phi_1, \Phi_2, \ldots$ is monotone increasing.

The first fact we need is that $w_2(t+n(k+1)) \ge w_2(t+n(k))$ for $t \in [0, n(1))$. This we show as follows.

$$w_2(t+n(k+1)) = e^{-\alpha(t+n(k+1))}\Big[\sum_{j=0}^{k} \lambda_{2j} \int_{n(j)}^{n(j+1)} e^{\alpha\xi}(\beta w_1(\xi-\tau)+I_2(\xi))\,d\xi$$

$$+ \int_{n(k+1)}^{t+n(k+1)} e^{\alpha\xi}(\beta w_1(\xi-\tau)+I_2(\xi))\,d\xi\Big]$$

$$\geq e^{-\alpha(t+n(k+1))}\Big[\sum_{j=0}^{k-1} \lambda_{2,j+1} \int_{n(j)}^{n(j+1)} e^{\alpha(\xi+n(1))} \cdot$$

$$(\beta w_1(\xi+n(1)-\tau)+I_2(\xi+n(1)))\,d\xi$$

$$+ \lambda_{2,k+1} \int_{n(k)}^{t+n(k)} e^{\alpha(\xi+n(1))}(\beta w_1(\xi+n(1)-\tau)+I_2(\xi+n(1)))\,d\xi\Big].$$

The inequalities $w_1(\xi + n(1) - \tau) \geq w_1(\xi - \tau)$ and $I_2(\xi + n(1)) \geq I_2(\xi)$ imply

$$w_2(t+n(k+1)) \geq e^{-\alpha(t+n(k))}\Big[\sum_{j=0}^{k-1} \lambda_{2,j+1} \int_{n(j)}^{n(j+1)} e^{\alpha\xi} \cdot$$

$$(\beta w_1(\xi-\tau)+I_2(\xi))\,d\xi$$

$$+ \lambda_{2,k+1} \int_{n(k)}^{t+n(k)} e^{\alpha\xi}(\beta w_1(\xi-\tau)+I_2(\xi))\,d\xi\Big].$$

Consider the coefficients λ_{2j} appearing in the right hand side of this inequality. Since

$$\lambda_{2j} = D_2(j)/(D_2(j)+(n-2)D_3(j))$$

$$= \frac{D_2(j)/D_3(j)}{D_2(j)/D_3(j) \quad +(n-2)},$$

λ_{2j} is monotone increasing in j if $D_2(j)/D_3(j)$ is monotone increasing in j. $D_2(j)/D_3(j)$ is obviously monotone increasing by (2. 27), since the coefficients of the summands in this series are nonnegative. Since λ_{2j} is monotone increasing in j, $\lambda_{2j} \geq \lambda_{2,j-1}$ and

$$w_2(t+n(k+1)) \geq e^{-\alpha(t+n(k))}\Big[\sum_{j=0}^{k-1} \lambda_{2j} \int_{n(j)}^{n(j+1)} e^{\alpha\xi} \cdot$$

$$(\beta w_1(\xi-\tau)+I_2(\xi))\,d\xi$$

$$+ \lambda_{2k} \int_{n(k)}^{t+n(k)} e^{\alpha\xi}(\beta w_1(\xi-\tau)+I_2(\xi))\,d\xi\Big]$$

$$= w_2(t+n(k)).$$

From the inequalities $w_2(t+n(k+1)) \geq w_2(t+n(k))$, $w_1(t+n(k+1) \geq w_1(t+n(k))$,

and $I_2(t+n(k+1)) \geq I_2(t+n(k))$, it readily follows by (2. 25) that the sequence $\Phi_1, \Phi_2, \ldots$ is monotone increasing.

PART III.

Derivation of

$$\lim_{i\to\infty} \frac{V_{i-1}+c}{V_{i-1}+c+\Psi_i S^{i-1}} > \frac{1}{S}$$

where $S = R^{-1} = e^{u(w+W)} > 1$.

Let

$$u_{i-1} \equiv S^{1-i} V_{i-1} = \sum_{k=1}^{i-1} (\Psi_k + \Phi_k) R^{i-k} .$$

We will show that

$$\lim_{i\to\infty} \frac{u_{i-1}}{1-S^{-i+1}} = \frac{\Phi+\Psi}{S-1} ,$$

and thus that

$$\lim_{i\to\infty} \frac{V_{i-1}}{S^{i-1}-1} = \frac{\Phi+\Psi}{S-1} . \tag{A. 7}$$

Since

$$\left| \frac{V_{i-1}+c}{\frac{\Phi+\Psi}{S-1}(S^{i-1}-1)+c} - 1 \right| = \left| \frac{V_{i-1}(S-1)}{(S^{i-1}-1)(\Phi+\Psi)} - 1 \right| \left(\frac{\frac{(\Phi+\Psi)(S^{i-1}-1)}{S-1}}{\frac{(\Phi+\Psi)(S^{i-1}-1)}{S-1}+c} \right)$$

$$\leq \left| \frac{V_{i-1}}{S^{i-1}-1} - \frac{\Phi+\Psi}{S-1} \right| \left(\frac{S-1}{\Phi+\Psi} \right) ,$$

(A.7) implies $\lim_{i\to\infty}(V_{i-1}+c)(\frac{\Phi+\Psi}{S-1}(S^{i-1}-1)+c)^{-1}=1$. Similarly one can show that

$$\lim_{i\to\infty}\left(V_{i-1}+c+\Psi_i S^{i-1}\right)\left(\frac{\Phi+\Psi}{S-1}\left(S^{i-1}-1\right)+c+\Psi S^{i-1}\right)^{-1}=1.$$

Dividing these two equations then gives

$$\lim_{i\to\infty}\frac{V_{i-1}+c}{V_{i-1}+c+\Psi_i S^{i-1}}\;\frac{\frac{\Phi+\Psi}{S-1}\left(S^{i-1}-1\right)+c+\Psi S^{i-1}}{\frac{\Phi+\Psi}{S-1}\left(S^{i-1}-1\right)+c}=1.$$

But

$$\lim_{i\to\infty}\frac{\frac{\Phi+\Psi}{S-1}\left(S^{i-1}-1\right)+c+\Psi S^{i-1}}{\frac{\Phi+\Psi}{S-1}\left(S^{i-1}-1\right)+c}=\frac{\frac{\Phi+\Psi}{S-1}+\Psi}{\frac{\Phi+\Psi}{S-1}}$$

$$=\frac{\Phi+S\Psi}{\Phi+\Psi}$$

since $S>1$. Thus we will be able to conclude that

$$\lim_{i\to\infty}\frac{V_{i-1}+c}{V_{i-1}+c+\Psi_i S^{i-1}}=\frac{\Phi+\Psi}{\Phi+S\Psi}$$

$$>\frac{1}{S}$$

by the positivity of Φ and Ψ, which will complete the proof.

It remains only to show that

$$\lim_{i\to\infty}\frac{U_{i-1}}{1-S^{-i+1}}=\frac{\Phi+\Psi}{S-1},$$

where $U_{i-1}=\sum_{k=1}^{i-1}(\Psi_k+\Phi_k)R^{i-k}$. Let $Q(i)=\Psi_i+\Phi_i$ and $Q=\Psi+\Phi$.

Then $\lim_{i\to\infty}Q(i)=Q$. Consider

$$Q\frac{1-S^{-i+1}}{1-S}-U_{i-1}=Q\frac{1-S^{-i+1}}{S-1}-\sum_{k=1}^{i-1}\left(\Psi_k+\Phi_k\right)R^{i-k}$$

$$= \sum_{k=1}^{i-1} Q R^{i-k} - \sum_{k=1}^{i-1} Q(k) R^{i-k}$$

$$= \sum_{k=1}^{\langle i \rangle} (Q - Q(k)) R^{i-k} + \sum_{k=\langle i \rangle+1}^{i-1} (Q - Q(k)) R^{i-k},$$

where

$$\langle i \rangle = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even} \\ \frac{i+1}{2} & \text{if } i \text{ is odd} . \end{cases}$$

Given $\epsilon > 0$, the inequality

$$\max\left(R^{i-\langle i \rangle}, Q - Q(\langle i \rangle)\right) \leq \frac{1-R}{1+2Q}\,\epsilon$$

holds for all sufficiently large i. Thus for a fixed large i and all $j \geq \langle i \rangle$,

$$Q - Q(j) \leq \frac{1-R}{1+2Q}\,\epsilon$$

since $\Psi_1, \Psi_2, \ldots$ and $\Phi_1, \Phi_2, \ldots$ are monotone increasing sequences. From this follows

$$\left| Q \frac{1 - S^{-i+1}}{S-1} - U_{i-1} \right| \leq \sum_{k=1}^{\langle i \rangle} (Q + Q(k)) R^{i-k} + \sum_{k=\langle i \rangle+1}^{i-1} |Q - Q(k)| R^{i-k}$$

$$\leq 2QR^{i-\langle i \rangle} \sum_{k=1}^{\langle i \rangle} R^{\langle i \rangle - k} + \frac{1-R}{1+2Q}\,\epsilon \sum_{k=\langle i \rangle+1}^{i-1} R^{i-k}$$

$$\leq 2Q \frac{1-R}{1+2Q}\,\epsilon \sum_{k=0}^{\infty} R^k + \frac{1-R}{1+2Q}\,\epsilon \sum_{k=0}^{\infty} R^k$$

$$= \frac{2Q}{1+2Q}\epsilon + \frac{1}{1+2Q}\epsilon$$

$$= \epsilon$$

or

$$\lim_{i\to\infty}\left(Q\,\frac{1-S^{-i+1}}{S-1} - u_{i-1}\right) = 0,$$

and

$$\lim_{i\to\infty}\left(\frac{u_{i-1}(S-1)}{Q(1-S^{-i+1})} - 1\right)\frac{Q}{S-1}\left(1-S^{-i+1}\right) = 0 .$$

Since $S > 1$, $\lim_{i\to\infty}(1-S^{-i+1}) = 1$. Thus

$$\lim_{i\to\infty}\frac{u_{i-1}}{1-S^{-i+1}}\,\frac{(S-1)}{Q} = 1,$$

which is the same as

$$\lim_{i\to\infty}\frac{u_{i-1}}{1-S^{-i+1}} = \frac{\Phi+\Psi}{S-1} .$$

The proof is therefore complete.

APPENDIX B

COMPUTER DATA FOR COMPLETE 3-GRAPHS WITH LOOPS

Some computer data have been gathered for the system with coefficient matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

and $\tau = 0$. This has been kindly done by P. R. Stein and J. Neergaard at Los Alamos. For the case $\tau = 0$, we have Theorem 3.1 at our disposal. Several graphs of this case for different choices of initial data of the ratios X_i and y_{jk} are illustrated in Figures 26, 27, and 28. Two cases with $\tau = 1$ are shown in Figures 29 and 30. Six cases with $\tau = 2$ are given in Figures 31 through 37.

When $\tau = 0$, Theorem 3.1 tells us that $\dot{y}_{ij}$ can change sign at most once. In the Figures 29-37 with $\tau > 0$, we see that this is not a general result since several of the $\dot{y}_{ij}$ s change sign at least twice. Nonetheless, the same limits are approached for these particular choices of $\tau > 0$, suggesting that there is a range of positive τ for which the same limits are approached. This suggestion is compatible with Theorem 4.1 for the variational system associated with this P.

In these graphs Y(1) stands for y_{12}, Y(2) for y_{21}, and Y(3) for y_{31}, while X(i) stands for X_i, i = 1, 2, 3.

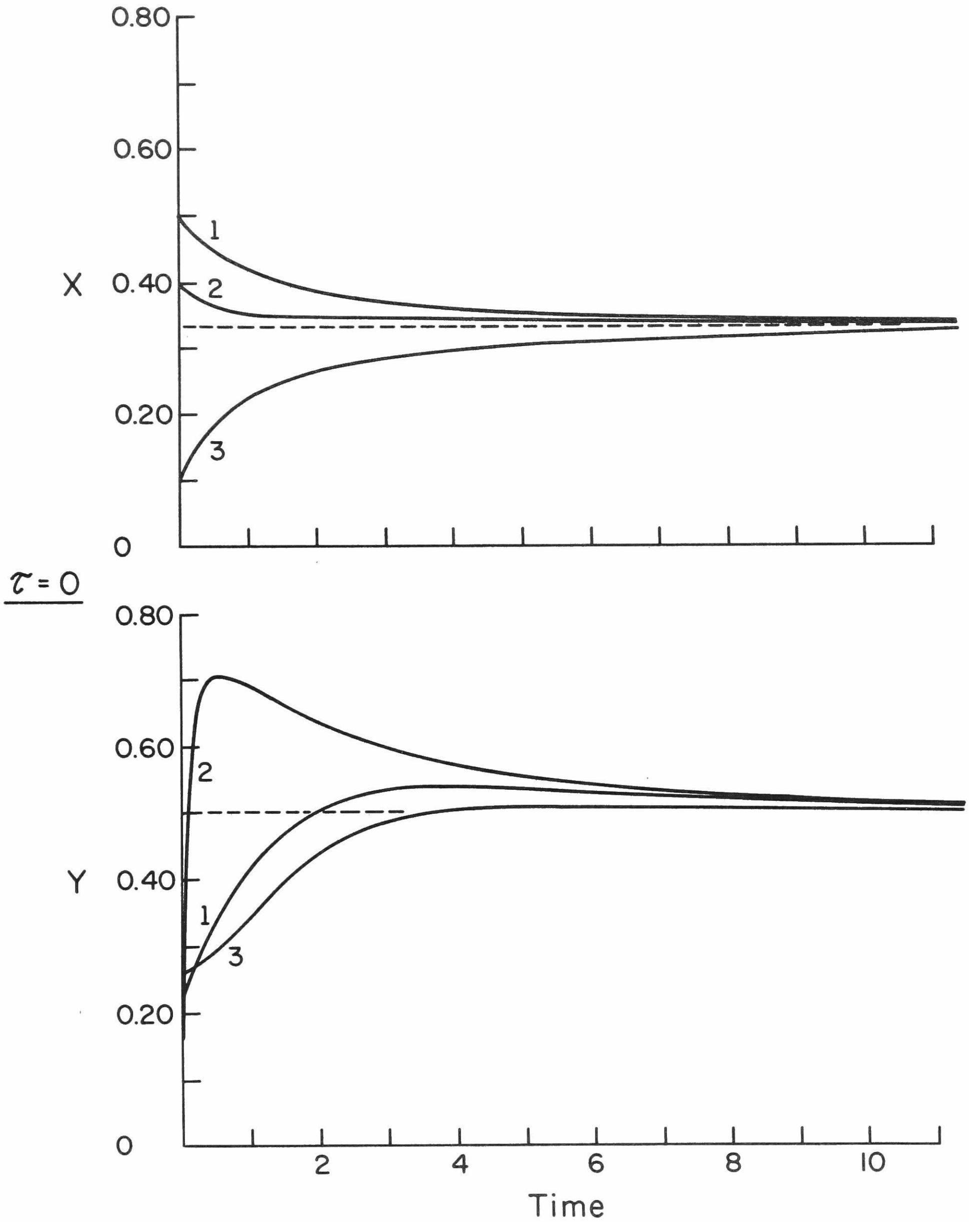


Fig.26

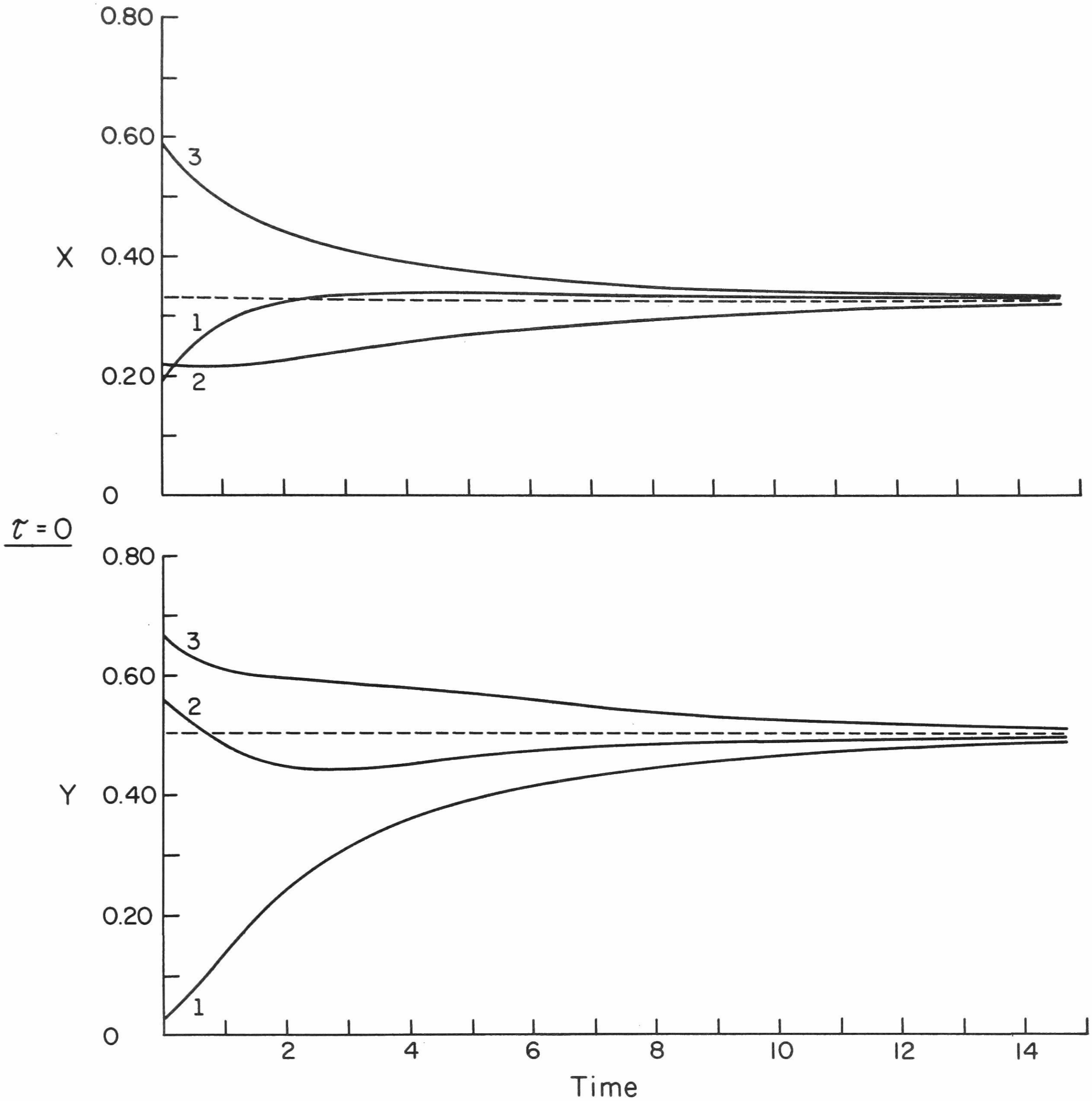

0.80
0.60
X 0.40
0.20
0
3
1
2
$\tau = 0$
0.80
0.60
Y 0.40
0.20
0
3
2
1
2
4
6
8
10
12
14
Time

Fig. 27

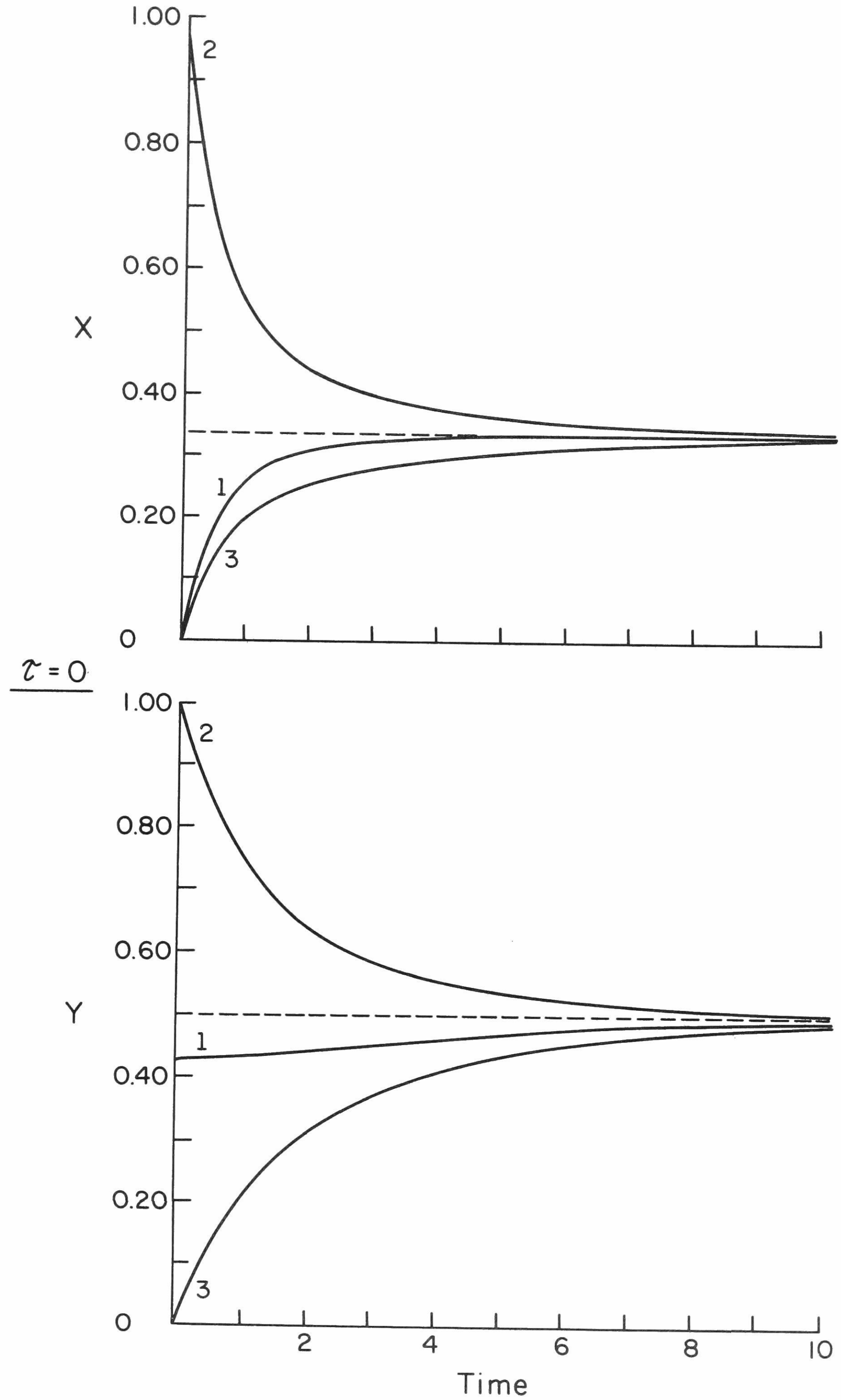

1.00
0.80
0.60
X
0.40
0.20
0
2
1
3
$\tau = 0$
1.00
0.80
0.60
Y
0.40
0.20
0
2
1
3
2
4
6
8
10
Time

Fig. 28

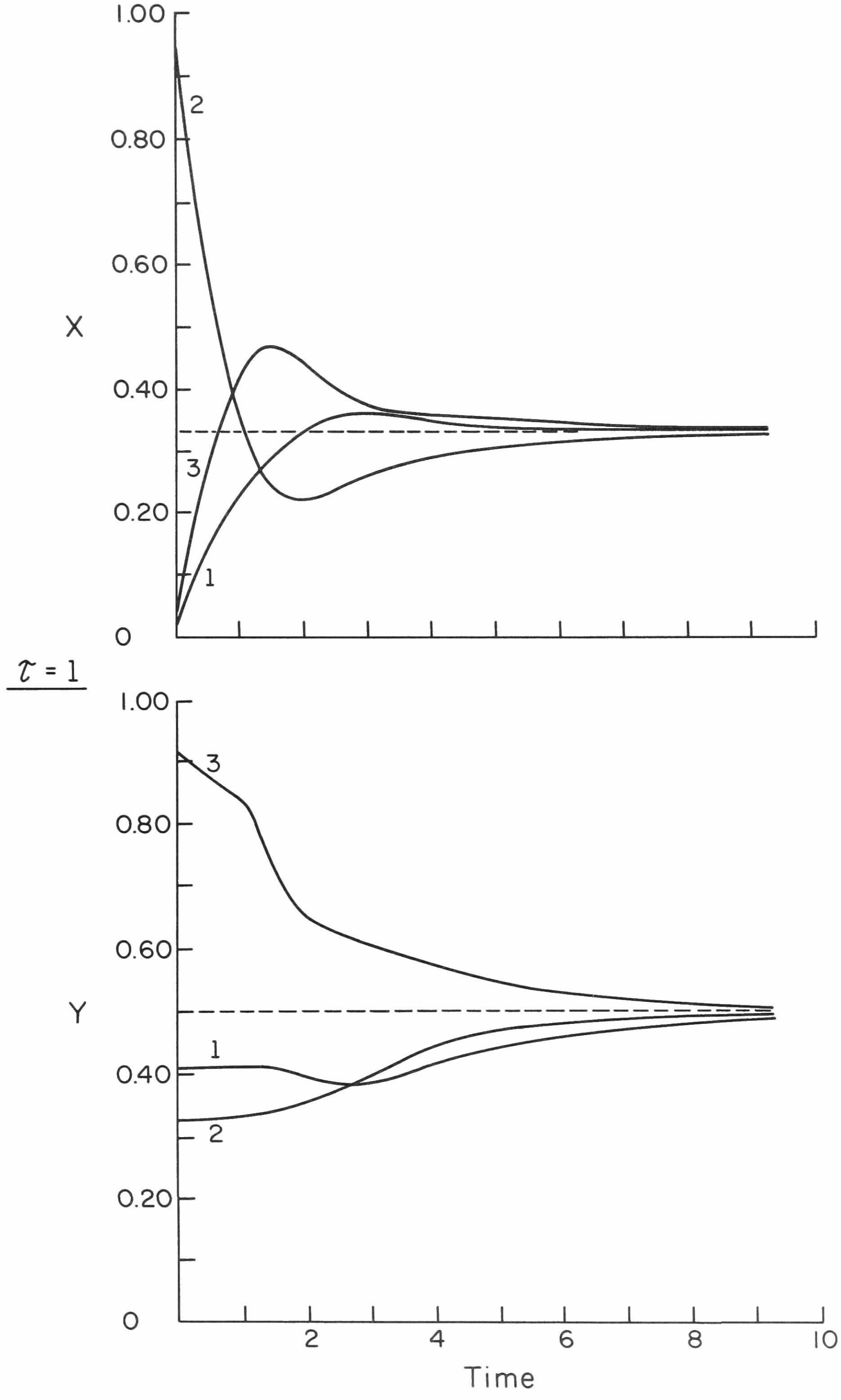


Fig. 29

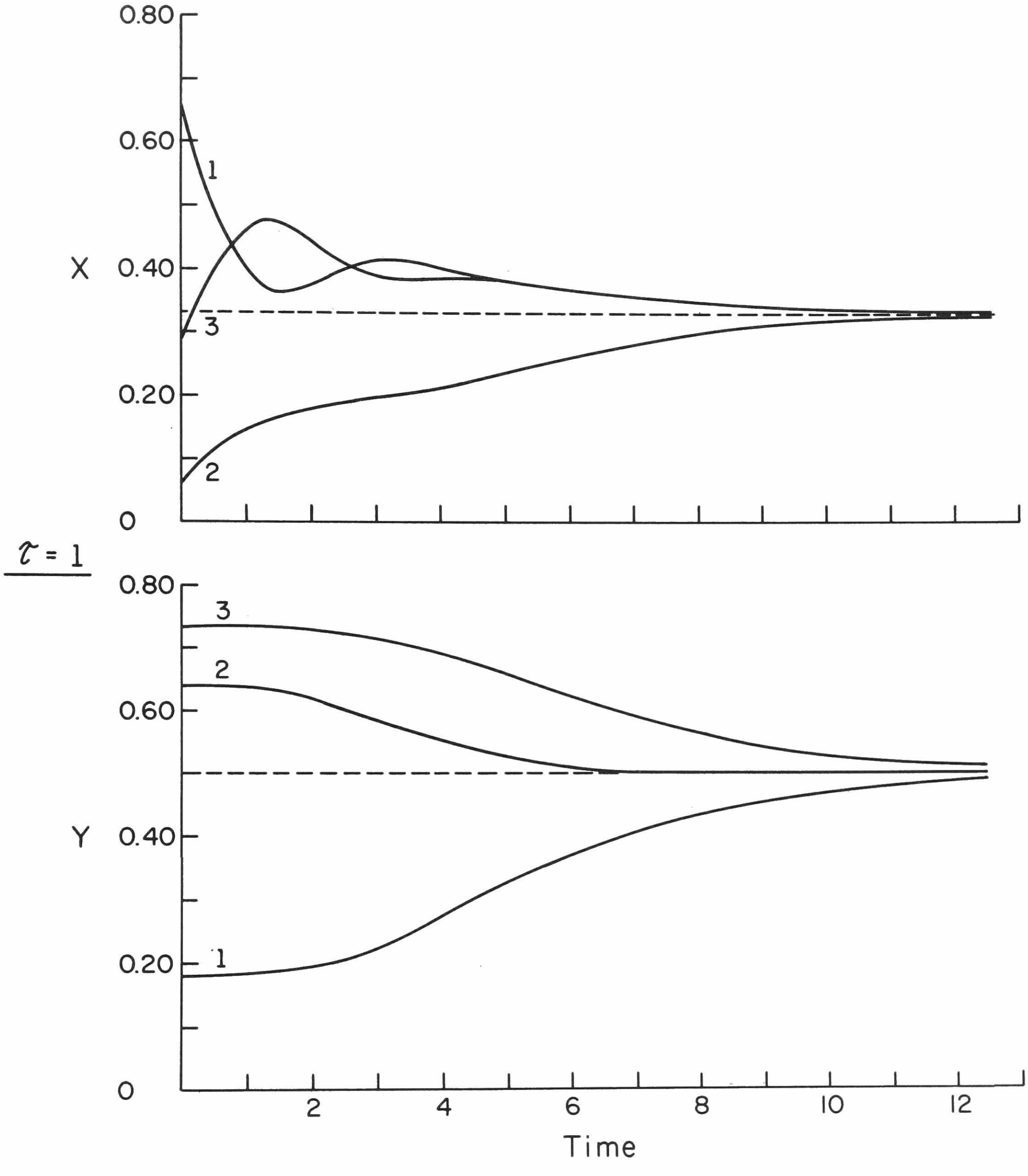

0.80
0.60
X 0.40
0.20
0
1
2
3
$\tau = 1$
0.80
0.60
Y 0.40
0.20
0
3
2
1
2
4
6
8
10
12
Time

Fig. 30

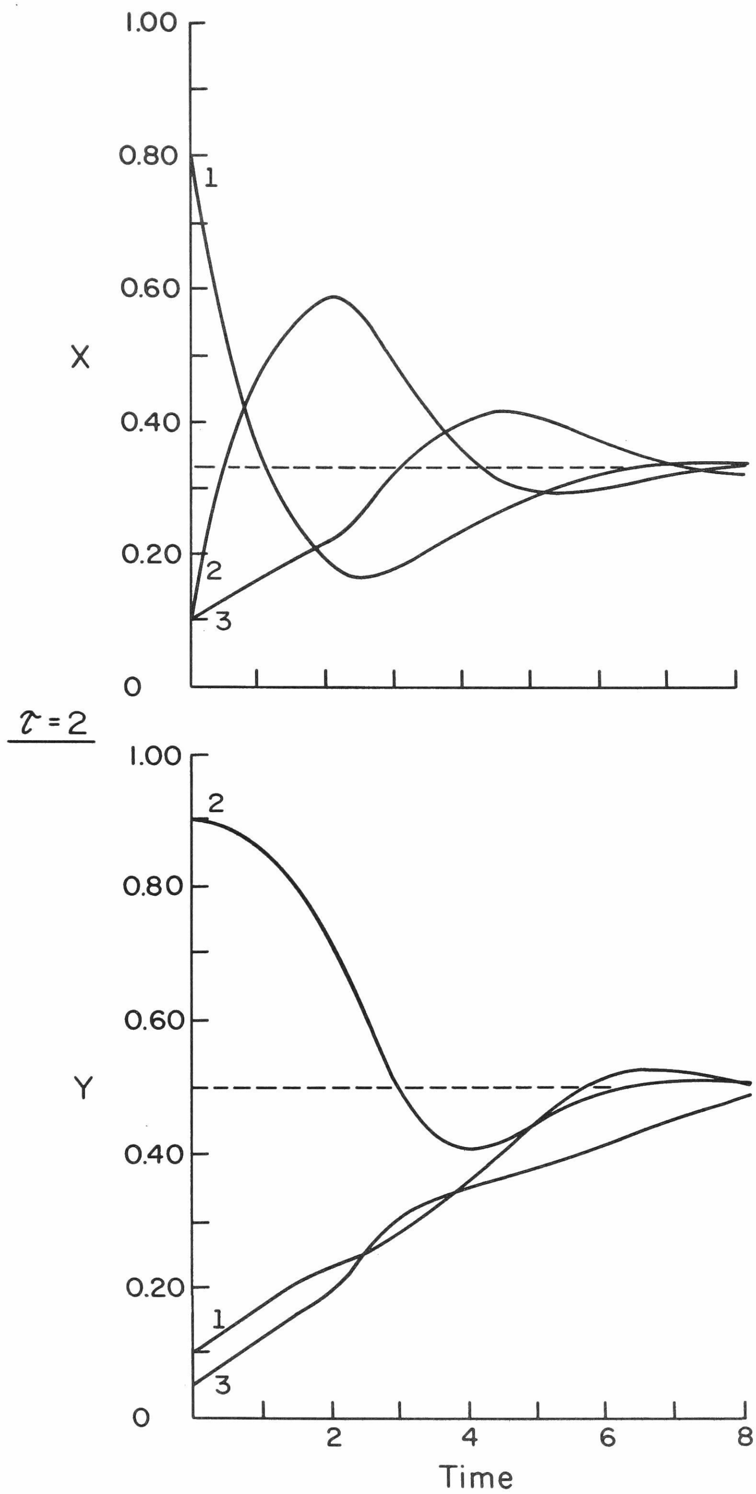

1.00
0.80
0.60
0.40
0.20
0
X
1
2
3
τ = 2
1.00
0.80
0.60
0.40
0.20
0
Y
2
1
3
2
4
6
8
Time

Fig. 31

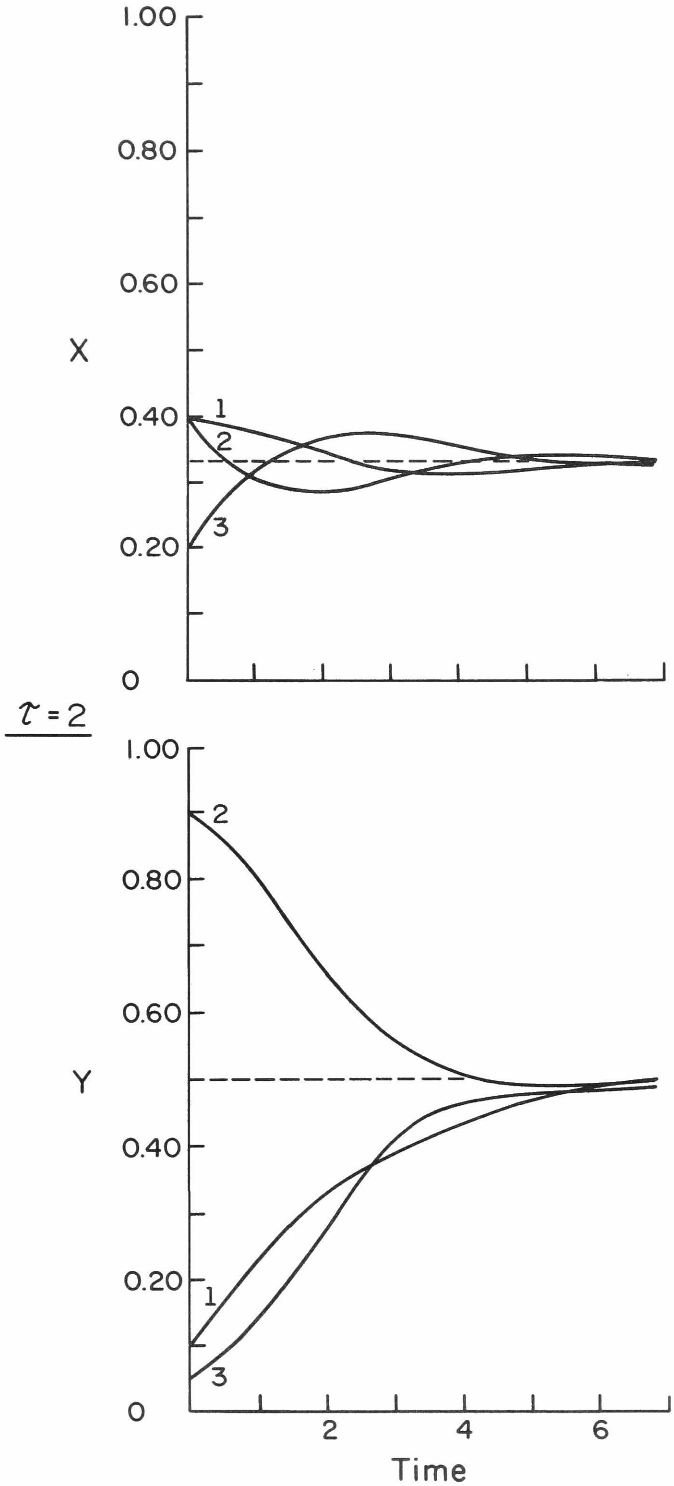

1.00
0.80
0.60
X
0.40
0.20
0
1
2
3
τ = 2
1.00
0.80
0.60
Y
0.40
0.20
0
2
1
3
2
4
6
Time

Fig. 32

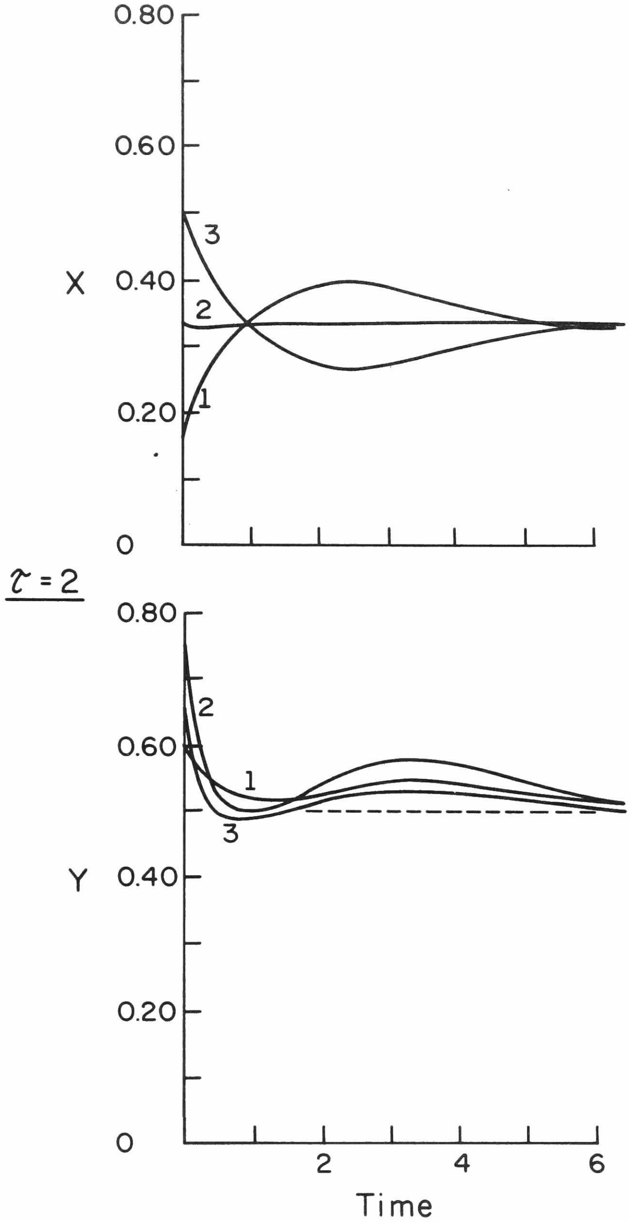


Fig.33

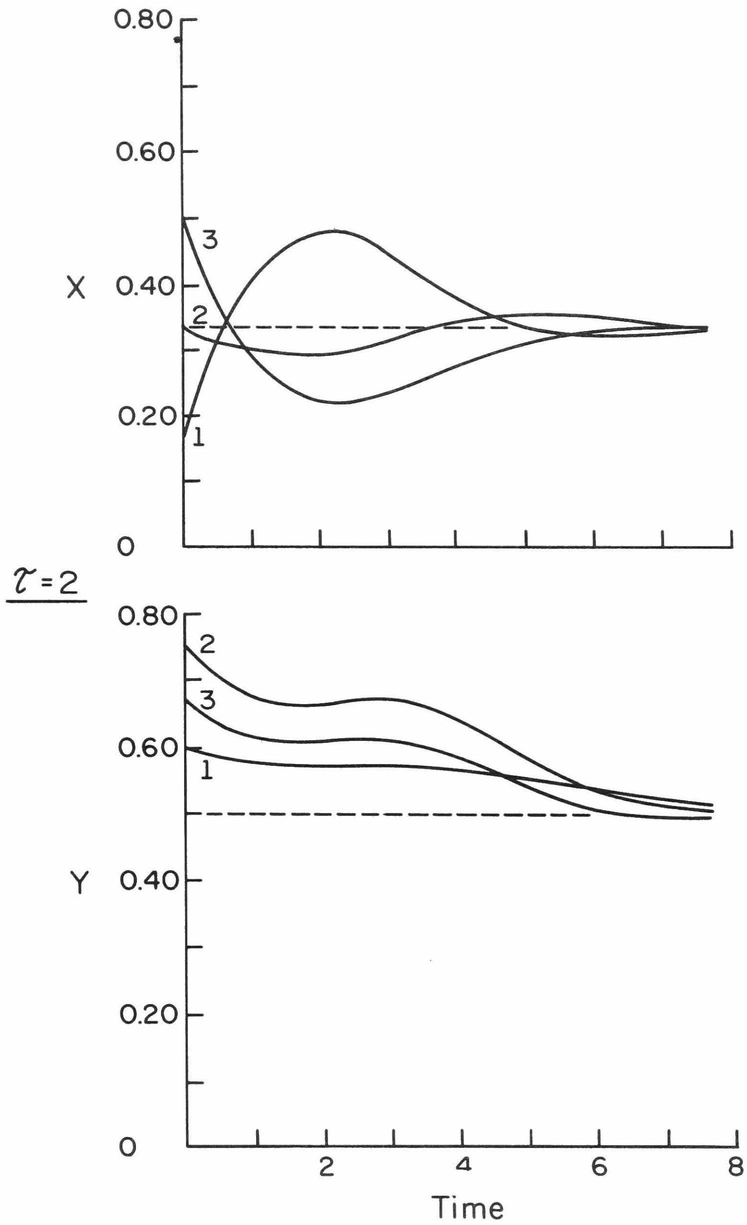


Fig.34

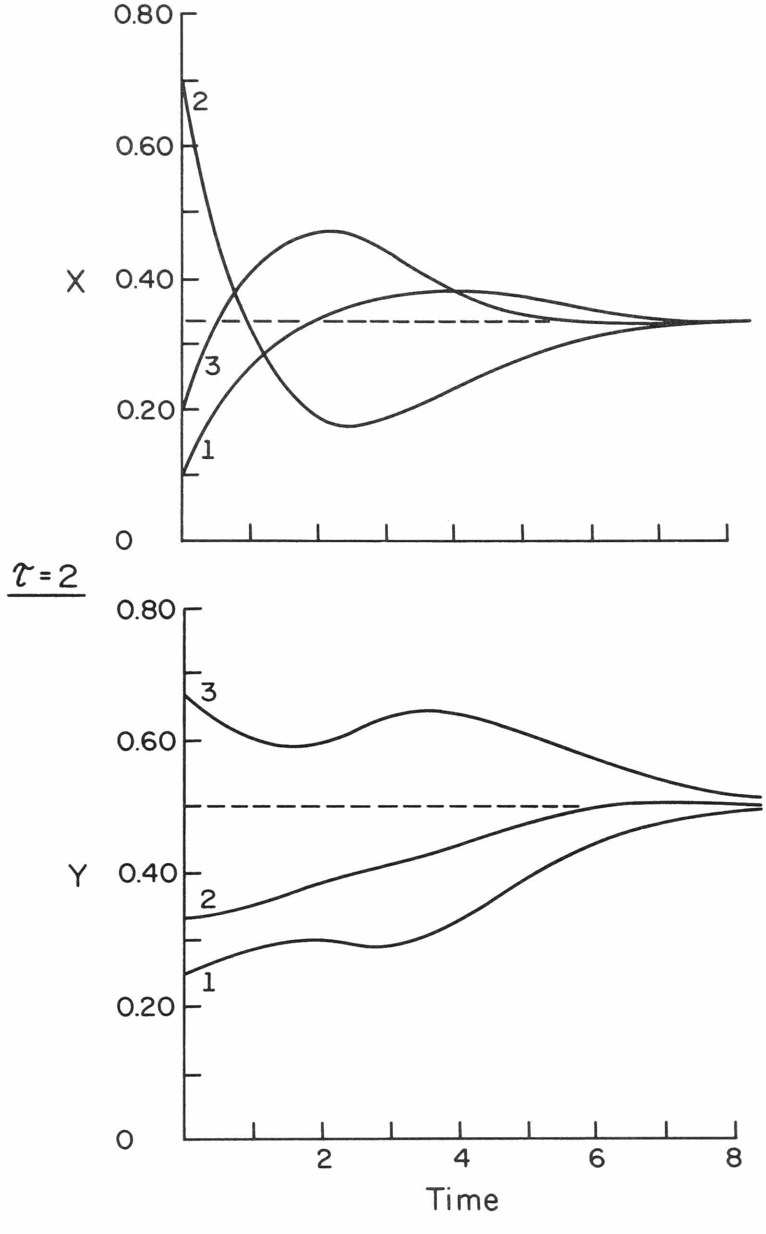


Fig. 35

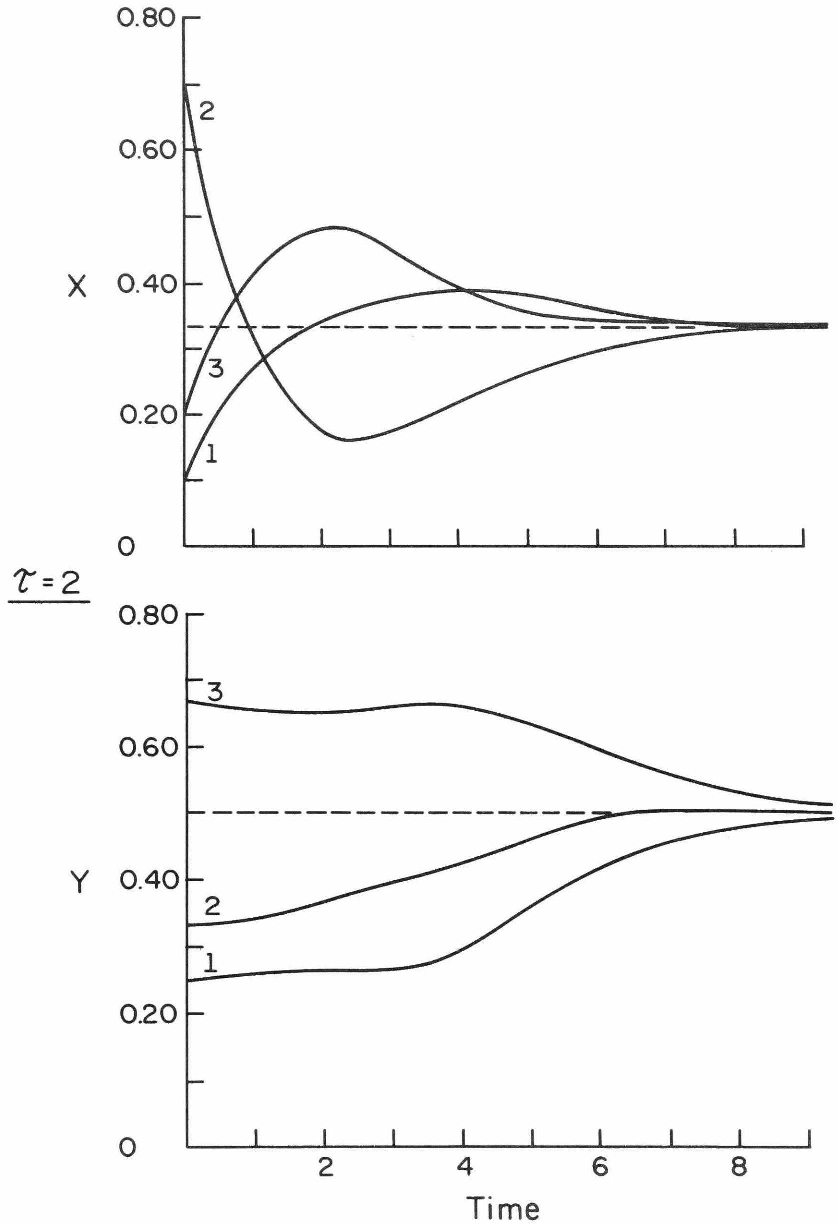


Fig. 36

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