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Second Order Logic and Logical Form

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SECOND ORDER LOGIC AND LOGICAL FORM

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INTRODUCTION

This thesis explores several related issues surrounding second order logic. The central problem running throughout is whether second order logic should provide the underlying logic for formalizations of natural language. A prior problem is determining the significance of this choice.

Such controversies over the adoption of a logic usually involve assessing the merits of challengers to first order logic. In some of these rival systems various first order logical truths do not hold. The failure of the Law of the Excluded Middle in intuitionistic systems is the most common example. The other alternatives to first order logic accept it as a part of the truth, but extend it by adding new logical constants. Some modal systems of logic are formed by adding to first order logic a symbol intended to be read as 'it is logically necessary that.' The first order semantics is extended to provide truth conditions for sentences containing this new symbol. In such cases the debate is whether we are justified in expanding the list of logical constants provided by first order logic. We accept the first order logical constants and are deciding whether, e.g., 'it is logically necessary that' should be added to the list.

Second order logic is such an extension of first order logic, treating quantification over properties as a logical constant. There are individual variables to range over objects in the domain of an interpretation and predicate variables to range over properties of those objects. The formal semantics is provided in set theory. An interpretation of a second order language is identical with an interpretation of its first order sublanguage, with the predicate variables ranging over the powerset of the domain. If P is a predicate variable and x is an individual variable, then the formula Px is satisfied, relative to an assignment to the variables of the language, just in case the element of the domain assigned to x is a member of the subset of the domain assigned to P. Other formulas containing predicate variables

have the satisfaction conditions one would expect. Our main concern is whether first order logic should be extended in this way.

In Chapter II we will describe a second order language and its semantics in some detail, but first the intuitive notion of property employed in second order logic must be clear. This is the notion of properties as arbitrary collections. Given a totality T, what are the properties of the elements of T as construed by second order logic? These properties satisfy the following two claims. First, for each arbitrary collection of elements of T there is a property possessed by precisely those objects. Second, no two distinct properties are each possessed by the same objects, that is, properties are extensional. For any totality T, the properties of members of T correspond precisely to the subtotalities of T and an object possesses a property if and only if it is a member of the corresponding subtotality. Accordingly, we identify the properties with the subtotalities or arbitrary collections of elements of T and identify possessing a property with membership. If T is a set, then the properties we quantify over in second order logic are just the subsets of T.

We must keep this conception of property in mind so that our discussion of second order logic is not obscured by considerations only relevant to other conceptions. Most well-known are intensional notions, where distinct properties may be true of the same objects. Quine has argued extensively that intensional properties are unacceptable since they lack a criterion of individuation. One holding this view might criticize second order logic on the grounds that quantification over properties is unclear, but this would be a mistake. In second order logic properties are not intensional. Like sets, properties conceived as arbitrary collections are individuated by an extensionality criterion.

Similarly, second order logic does not receive support if some other notion of property proves useful. Lately there has been interest in the view that there does not always exist a property true of each member of an arbitrary collection. We feel that such objects may not

have anything in common and that only certain collections can be defined by what we could call a <u>real</u> property. Many explicate this concept by providing a theory of physical properties, although other versions are possible. Beyond appeals to intuition, such conceptions of property are defended by their usefulness in solving various philosophical problems. For example, a satisfactory concept of physical property seems to make possible an account of natural laws. Belief in the need for quantification over real properties of some sort may tempt one to cite this in defense of second order logic, but the philosophical importance of some notion of real property is irrelevant to the choice between first and second order logic.

The nature of logical theory is the topic of Chapter I. We argue that the two problems for a logical theory are to provide a precise account of logical consequence through a theory of logical form and to characterize inference. Modern logical theories attempt to solve these problems by translating natural language into a formal language for which we have precise characterizations of logical form and inference.

In Chapter II we describe a system L2 of second order logic and establish some simple relationships between second and first order languages. We show that given any second order theory there is a corresponding first order theory in which the same assertions can be expressed and proven. First and second order languages differ in their account of logical form as expressed in their definitions of truth under an interpretation.

Chapter III examines Frege's logical theories and explains why the first modern systems of logic were second order. The explanation is not that Frege simply thought set theory was part of logic. Beyond the error in identifying second order logic with set theory, we show that Frege had a good justification for his second order systems based on his analysis of language into the application of function to argument.

Some feel that the lack of a complete proof procedure for a logic is a reason to reject it. Although the most common versions of this view are not correct, in Chapter IV we show that the existence of a recognizably complete proof procedure for first order logic shows that it avoids certain undesirable epistemological consequences of second order logic. These undesirable consequences are the basis of an argument against second order logic, but this argument assumes controversial views concerning scientific methodology and the limits of human knowledge.

In Chapter V we present a different criticism of second order logic. We argue that our ability to quantify over ever larger totalities shows that any collection may be treated as an object. Quantification over the powerset of a set is never correctly formalized as second order. Logical theories based on second order logic thus provide an incorrect account of logical form, and we show that other higher order logics are subject to the same criticism.

CHAPTER I

Before discussing the specific controversy surrounding second order logic, we consider the general problem of the nature of a logical theory. What are the problems a logical theory attempts to solve? Why have logical theories taken the form they have? These questions are intrinsically interesting, and our answers will prove useful in the following chapters. The main tasks of a logical theory are seen to be providing precise accounts of the relations of logical consequence and inference. We characterize our informal concept of these relations and show how modern logical theories attempt to explicate them.

We start by considering arguments. Typically, a person presents an argument in order to establish the truth of its conclusion. The other sentences of the argument are intended to provide support for this conclusion. These sentences are either premisses, to be accepted on the basis of common knowledge, or else they are supported by other sentences in the argument. In the latter case, the argument is complex, containing an argument as a proper part. We can confine our attention to simple arguments in which all sentences other than the conclusion are premisses.

In a good argument the truth of the premisses makes the conclusion more probable. In some arguments, however, the relation between the premisses and the conclusion is particularly strong, for the truth of the premisses--in some sense we want to explicate--<u>guarantees</u> the truth of the conclusion. For example, the truth of (1) 'If Sam is a politician, then Sam is unscrupulous' and (2) 'Sam is a politician' guarantees that (3) 'Sam is unscrupulous' is also true.

This example has another interesting property. The relation between the premisses and the conclusion, whereby the truth of the premisses guarantees the truth of the conclusion, seems to depend only on the logical form of the sentences involved and not on their particular content. The truth value of a sentence clearly depends on more than

just the meaning of its significant parts. 'Sam hit Slim' may be true while 'Slim hit Sam' is false. The structure of the sentence is also relevant. The logical form is the structure which, together with the meaning of the significant parts of a sentence, determines its truth value. In our example, the relation whereby the truth of (1) and (2) guarantee the truth of (3) is only sensitive to the logical form of these sentences. That they are about Sam, politicians and lack of scruples is unimportant. Only their logical form is relevant. This relation also holds if the sentences are 'If John is an acrobat, then John is agile', 'John is an acrobat, John is agile'; 'If Bob is a cat burglar, then there are not enough police,' 'Bob is a cat burglar,' 'There are not enough police'; and so on. These groups of sentences all seem to have a certain structure in common which we might represent as 'If A, then B,' 'A,' 'B.' Considering further examples will quickly convince one that in any argument with this general form the truth of the premisses guarantees the truth of the conclusion.

When the truth of each member of a set X of sentences guarantees the truth of a sentence A solely by virtue of their logical form, we say that A is a _logical consequence of_ X. In the arguments we have considered the conclusion was a logical consequence of the set of premisses. Such arguments are thus different from those in which the truth of the conclusion is guaranteed by the truth of the premisses, but where more than the form of the sentences is involved. For instance, one might argue that the truth of 'John is a bachelor' guarantees the truth of 'John is male,' due to the meaning of 'bachelor.' Yet this relation is not formal. In constructing a logical theory we are interested in characterizing the (formal) relation of logical consequence.

One of the two fundamental problems facing a logical theory is to provide a precise characterization of the relation of logical consequence. Given our informal concept of this relation, an acceptable account of logical consequence must at least meet the following demands. First, it must characterize a consequence relation between sets $X = \{x_i : i \in I\}$ of sentences and single sentences A satisfying the condition

that if all the x_i are true then so is A. The observation that the relation of logical consequence holds solely in virtue of the logical form of the sentences involved justifies the second demand. An account of logical consequence must provide a theory of logical form and show that the consequence relation is only sensitive to the logical form of sentences. An account of logical form must specify a vocabulary and a stock of constructions adequate to generate all sentences of the language and show how the truth value of each sentence results from its particular composition from elements of the vocabulary. An account of logical form thus is a theory of truth for the language.[1] For the relation R claimed to be the consequence relation, a logical theory must imply that if R(X,A) and if $A', X' = \{x_i' : i \in I\}$ are such that A' and each x_i' have the same logical form as A and x_i, then R(X',A').

Of course, any logical theory which fulfills these conditions does not thereby provide a correct account of logical consequence. To take an extreme case, a logical theory could define R to be the empty relation. R then trivially satisfies these conditions, whatever our theory of logical form. But to be acceptable, a characterization of logical consequence must cohere with our intuitions. As usual, if we have good reason we will conclude that some of our intuitions are mistaken. If a logical theory agrees with virtually none of our intuitions, however, then the concept of form and the relation of consequence it defines are not the concept of logical form and the relation of logical consequence in which we are interested.[2]

Before the invention of formal languages, there were tremendous obstacles to providing a theory of logical form for natural language. The principal problem was the bewildering variety of logically equivalent constructions in natural language. For example, we feel that 'if A, then B' has the same logical form as 'if A, B,' 'A, only if B,' and so on. Finding a way to provide a general account of the logical form of the sentences of natural language might appear to be a hopeless task. This difficulty received a brilliant solution with the invention of formal

languages. In a formal language we have a precise specification of the primitive symbols and of the constructions by which complex expressions are formed. We then specify how each construction contributes to the truth value or denotation of complex expressions in which it occurs. Finally, we frame a definition of logical consequence applying directly to sentences of this formal language.

Our goal is to find a formal language into which we can translate the sentences of natural language in such a way that it provides a model of logical form and consequence for natural language. For a formal sentence A to be a translation, or formalization, of a natural language sentence B they must have the same truth conditions.[3] By identifying the logical form and consequence relation of natural language with that projected on it by a formalization into a formal language, we get an account of logical form and logical consequence in natural language. If this identification is correct, we say that this is a correct formalization of natural language. If A', x_i' are correct formalizations of A, x_i, then A', x_i' have the same logical form as A, x_i and A' is a logical consequence of $X' = \{x_i' : i \epsilon I\}$ just in case A is a logical consequence of $X = \{x_i : i \epsilon I\}$. Our aim is to find a formal language permitting a correct formalization of natural language.

Some formal languages do not contain formalization of certain sentences of natural language. For example, no first order language contains a formalization of 'It is logically necessary that 2+5 = 7.' More relevant to our concern with second order logic, there may be formalizations of a natural language sentence in a formal language, but no correct formalization. If (4) 'There is a set of natural numbers containing 0' is valid, i.e., a logical consequence of the empty set, then no sentence of a first order language correctly formalizes it. There are first order sentences with the same truth conditions as (4), but they are nonlogical truths.

Since Tarski's seminal work on model theory, definitions of logical consequence for formal languages have employed the notions of an

interpretation of the language and of the truth of a sentence under an interpretation. An interpretation of a formal language L is an ordered pair $\langle D,F \rangle$, where D is a domain for the individual variables of L to range over and F is a function whose domain is a subset of the primitive symbols of L.[4] The symbols in the domain of F are the nonlogical symbols of L, usually the relation symbols, functions symbols and constants. F assigns a meaning to each of these symbols, that is, a set of n-tuples of members of D to each n-place relation symbol, an n-place function from D into D to each n-place function symbol and a member of D to each constant.

The definition of truth under an interpretation is a recursive definition of the truth conditions of each sentence relative to an interpretation of the language. Those constructions deemed constitutive of the logical form of sentences make the same contribution to the truth conditions of sentences in which they occur, regardless of the interpretation. The constant meaning of these constructions is specified in the basis and recursion clauses of the definition of truth under an interpretation. For example, the basis clause of the usual definition for first order languages says that a sentence of the form Gb is true under an interpretation $\langle D,F \rangle$ if and only if $F(b) \varepsilon F(G)$. The meaning of b and G can change from interpretation to interpretation, but predication is always interpreted as set membership. Similarly, in such definitions sentences of the form $A \wedge B$ are true under an interpretation I if and only if A is true under I and B is true under I. Thus, $\wedge$ is always interpreted as conjunction.

By providing an account of the interpretations of L and a recursive characterization of truth under an interpretation, we provide a specification of the logical form of the sentences in L. The sentences of L are generated from the primitive predicate symbols, function symbols and constants by the application of precisely specified constructions. For any interpretation of L, the definition of truth under an interpretation tells us the contribution these constructions make to determining the truth value of sentences in which they occur. We define the relation of

logical consequence for L by using the notion of truth under an interpretation. A sentence A is a logical consequence of a set X of sentences of L if and only if A is true under every interpretation under which all the members of X are true. A is valid, or a logical truth, if and only if A is a logical consequence of the empty set. Such definitions satisfy the requirement that logical consequence be a formal relation, that is, depend only on those features of sentences of L which the semantics deems part of their logical form. Since the meaning of the relation symbols, function symbols and constants varies from interpretation to interpretation, by considering all interpretations under which sentences are true we abstract from all but their logical form. The meaning of the nonlogical symbols is irrelevant, since the definition of logical consequence does not mention the actual interpretation of the language.

Although the usual definitions of logical consequence for formal languages accord with many of our intuitions, one may doubt whether such a definition can explicate our intuitive concept of logical consequence. An interpretation is a set and definitions of logical consequence are given in set theory, usually Zermelo-Fraenkel set theory with the Axiom of Choice. Prior to the discovery of Russell's Paradox, a strong comprehension principle for sets was assumed. This comprehension principle was

$$(\exists x)(\forall y)[y \varepsilon x \leftrightarrow Py] ,$$

where P stands for any property and the quantifiers range over all sets. That is, for any property there exists a set containing just the sets having this property. A second assumption was that any predicate definitely true or false of each individual set stands for a property of sets. From these assumptions the existence of the set of all sets follows by letting P stand for $y = y$. Russell's Paradox showed that these two assumptions are inconsistent. If we take P to be $y \notin y$, the comprehsnion principle assures us that there is a set x of all non-self-membered sets, but then $x \varepsilon x \leftrightarrow x \notin x$.

Two paths were open: one could retain the unrestricted comprehension principle but hold that some expressions seeming to express properties do not, e.g., because they are actually ill-formed; or one could abandon the unrestricted comprehension principle and replace it with one or more weaker principles. Russell took the first path in his theory of types. The unrestricted comprehension principle is retained, but restrictions on the means of expression prevent the formation of troublesome predicates like $y \notin y$. The second approach was Zermelo's. He replaced the unrestricted comprehension principle with the axiom of subsets: $(\forall z)(\exists x)(\forall y)[y \in x \leftrightarrow (y \in z \wedge Py)]$, where P stands for any definite property of sets. The effect is to maintain the comprehension principle for all properties expressible by predicates of the form $(y \in b) \wedge Py$, for some set b.

On both approaches, the collection of all sets is not itself a set. Since the domain of any interpretation is a set, under no interpretation do the individual variables of a formal language range over all sets. Two difficulties with standard set theoretic formulations of model theory follow. In set theory we seem to make statements about all sets. This view is controversial, and in the final chapter we will discuss an alternative to it. But if correct, the usual model theoretic semantics cannot provide an account of the meaning of set theoretic statements.

A related difficulty concerns model theoretic definitions of logical consequence. Georg Kreisel has argued[5] that when we say A is valid we don't mean merely that it is true under any interpretation as that notion is defined in set theory. We also mean that A is true when its variables range over collections that are not sets. If A contains a single binary predicate H, then we believe that it is true when its individual variables range over all sets and H is construed as the membership relation. Those interpretations satisfying a model theoretic definition might be called <u>set theoretic</u> interpretations, since the domain, relations and functions they provide are sets. Interpretations which are not sets, like the one we just gave to A, we will call <u>non-set theoretic</u>. Analogously, we can speak of set theoretic consequence and validity.

Kreisel's claim is that our intuitive concept of validity is truth under all interpretations, both set theoretic and non-set theoretic. Since set theoretic definitions of validity ignore non-set theoretic interpretations they fail to explicate our intuitive concept of validity.

Despite these defects in model theoretic definitions of validity, Kreisel shows that intuitive validity and set theoretic validity coincide in extension for first order formulas. Let Prov be the set of formulas provable in some standard proof procedure for the first order predicate calculus, SVal the set of set theoretically valid formulas and IVal the set of intuitively valid formulas, i.e., those true under all interpretations.[6] Restricting our attention to first order formulas, we have

$$\text{Prov} \subseteq \text{IVal},$$

since we can see that the axioms of standard proof procedures for first order logic are intuitively valid and that their rules of inference preserve intuitive validity. If a sentence is true under all interpretations, it clearly is true under all set theoretic interpretations, that is,

$$\text{IVal} \subseteq \text{SVal}.$$

Gödel's Completeness theorem established that

$$\text{SVal} \subseteq \text{Prov}.$$

Hence,

$$\text{SVal} = \text{IVal}.$$

Set theoretic validity and intuitive validity coincide in extension for first order formulas.

Characterizing the logical consequence relation is one major task of a logical theory. A second is providing an account of inference. We began this chapter by considering arguments in which the conclusion is a logical consequence of the premisses. Such arguments are called valid arguments or proofs and the conclusion is said to be inferred or proven from the premisses. We saw that in such arguments the truth of the premisses guarantees the truth of the conclusion in virtue of their logical

form. We then took this as our informal characterization of logical consequence: A is a logical consequence of a set X if and only if the truth of the members of X guarantees the truth of A in virtue of their logical form. We began by noting a feature of valid arguments but then characterized a semantic relation without reference to arguments at all.

The second fundamental problem facing a logical theory remains: characterize the inference relation. The conclusion of a valid argument is a logical consequence of the premisses. Beyond this necessary condition, however, there is a crucial epistemological component to our concept of a valid argument. Not only is the conclusion a consequence of the premisses, but one can come to know this. We believe that if one knows the premisses of a valid argument then one can come to know its conclusion by seeing that the argument demonstrates that the conclusion follows logically from the premisses. Valid arguments are tools we use to expand our knowledge.

Any acceptable account of inference must therefore meet two requirements. First, if A is the conclusion of a valid argument, then it must be a logical consequence of the premisses. Further, the argument must enable one to recognize this. This epistemological requirement yields, as an immediate consequence, a finiteness constraint on valid arguments. Since human beings have finite capacities, no one can draw a conclusion from an infinite number of premisses or follow infinitely many steps in a line of reasoning. Thus, any valid argument must be of finite length.

Since Frege's Begriffsschrift, attempted solutions to the inference problem have taken the form of providing a recursive proof procedure, or proof theory, for a formal language. A proof procedure consists of a set Ax of axioms and a set RI of rules of inference.[7] A proof procedure is recursive if Ax is recursive and RI is a finite set of recursive rules.[8] A formula A is provable from X if and only if there exists a finite sequence $A_1, \ldots, A_n$ such that (1) each A_i is either a member of $Ax \cup X$ or else results from the application of a rule in RI to earlier members of

the sequence and (2) $A_n = A$. Such a sequence is a proof. A formula provable from the empty set is simply said to be provable. The relation of being provable from a set and the concept of a formal proof are the explications provided of the inference relation and of the informal notion of a valid argument. As with formal characterizations of logical consequence, we get an account of inference in natural language through formalization.

An important assumption supports the view that any account of inference must take the form of a recursive proof procedure. This assumption is that only a recursively enumerable set of sentences can be known. It is plausible but has never been adequately defended. An appeal to intuition and Church's Thesis usually provide its support. In any rational reconstruction of our knowledge each sentence is either known directly or else recognized to be true by its relations to others known directly. Defenders of this view hold that the sentences known directly either must be finite in number or else be instances of a finite number of schemas, each known to have only true instances. Otherwise an infinite number of irreducibly different principles would be evident, a situation claimed to be impossible.[9] Similar intuitions support the claim that we come to know the remaining truths by seeing they follow from directly known truths by finitely many applications of a finite number of effective rules, each recognized to preserve truth. Only finitely many applications of these rules are permissible, since we can't follow infinite arguments. These rules must be finite in number, for infinitely many irreducibly different rules could not possibly be recognized to preserve truth. They must be effective or else we may be unable to recognize that a sentence follows from others by their means.

Church's Thesis links the informal concept of an effective rule to the formal concept of a recursive rule. It states that a function is effectively computable if and only if it is recursive. If we assume Church's Thesis, we can conclude that the knowable truths are generated by a finite number of recursive rules, each recognized to preserve truth.

Since the set of instances of a finite number of schemas is recursive, the set of sentences generated from such a set is recursively enumerable.

Adopting the assumption that the set of knowable truths is recursively enumerable, we can construe the inference problem as the problem of finding a recursive proof procedure generating the formalizations of all inferences. This is not the problem of finding a complete proof procedure. A complete proof procedure generates all instances of the consequence relation. A solution to the inference problem generates all recognizable instances of this relation. Though these problems are distinct, they are related as we shall discuss in Chapter IV.

If an appropriate proof procedure is chosen, it can be seen to satisfy our two conditions on solutions to the inference problem. If we choose axioms that are valid and rules of inference that yield logical consequences of sentences to which they are applied, then our proof procedure is sound. In a sound proof procedure the conclusion of a proof is clearly a logical consequence of the premisses. A proof procedure will satisfy the second condition if we can recognize its soundness. The axioms must be obviously valid and the rules of inference seen to yield only logical consequences of sentences to which they are applied. Assuming that all recursive functions are effectively computable, we can effectively recognize any proof. If we know the proof procedure is sound, we can see that the conclusion of any proof is a logical consequence of the premisses.

This proof theoretic approach to the inference problem employs idealizations common when providing rational reconstructions of our knowledge. The axioms in a proof procedure are all instances of a number of schemas. In such reconstructions we assume that we can effectively recognize all instances of a schema, whatever their complexity. We also assume that for any finite sequence, we can determine whether it is a proof using our effective procedure to see if each sentence is either an axiom, a premiss or the result of applying one of the rules of inference. These are both idealizations. People cannot actually comprehend sentences or

follow proofs of arbitrary finite length. Due to limitations on attention and speed of recognition, there is a least finite number n such that no sentence containing more than n symbols will ever be seen to be an instance of one of the axiom schemas of a formal system and such that no proof in such a system containing more than n symbols will ever be recognized. But since we can easily imagine possible histories of the world in which n would not have this property, e.g., due to people having greater attention spans, we assume that the actual value of n is not essential. If an instance of a schema or a proof contains more than n symbols, we feel that it is at least potentially recognizable.

On the other hand, we have required proofs to be of finite length. We feel that if people could comprehend proofs with infinitely many symbols human beings would be quite different creatures than they are. One might even believe that creatures who could comprehend infinite proofs would, for that very reason, not be human. Such considerations explain the common feeling that it is natural to exclude infinite proofs, while allowing proofs of arbitrary finite length.[10]

We have identified the explication of logical consequence and the explication of inference as the major tasks of a logical theory. Current logical theories first formalize natural language and then characterize these relations for this formal language. In the following chapters we consider the view that this formal language should be second order.

NOTES

1. Donald Davidson has written extensively on this theme. See, e.g., his 'On Saying That,' Synthese 19 (1968-9), pp. 131-3.

2. Michael Friedman makes a similar point concerning theoretical explications of concepts of physics in 'Grünbaum and the Conventionality of Geometry' in Space, Time and Geometry, ed. Patrick Suppes (Boston, 1973), pp. 231-2.

3. 'Truth conditions' will be given a precise meaning in the next chapter. Here it can be taken intuitively.

4. For simplicity, we only consider one-sorted languages.

5. 'Informal Rigour and Completeness Proofs' in The Philosophy of Mathematics, ed. Imre Lakatos (Amsterdam, 1972), pp. 152-5.

6. This terminology differs from Kreisel's.

7. In natural deduction systems a proof procedure consists solely of rules of inference.

8. When we speak simply of a 'proof procedure,' we will assume that it is recursive.

9. For example, Hilary Putnam says that he finds this supposition 'quite incredible.' See 'What Is Mathematical Truth?' in Mathematics, Matter and Method (London, 1975), p. 63.

10. There are difficult issues concerning such formal analyses of our informal concept of inference. When intuitively valid arguments in natural language are formalized, they rarely go over into formal proofs. Even if all the steps are intuitively evident, each will usually be represented by a number of applications of the formal rules of inference. One might try claiming that an argument in natural language is valid if the formalization of the conclusion can be formally proven from the formalization of the premisses. But if Goldbach's conjecture is formally provable from the first order Peano axioms, writing the English translations of a finite number of axioms sufficient for its proof and concluding 'Therefore, Goldbach's conjecture' would not constitute an informal proof. There are pragmatic constraints on how much of a formal proof may be omitted before we no longer have an informal proof.

Pragmatic constraints may also limit the complexity of informal proofs. In Remarks on the Foundations of Mathematics, trans. G.E.M. Anscombe (Cambridge, 1967), pp. 65ff and 83-4, Wittgenstein argues that a formal proof containing thousands of lines is not an informal proof, since it cannot be comprehended as a whole. Proof may even be context-dependent is a more radical sense. One could argue that in certain contexts $A_1, \ldots, A_n$, A constitutes an informal proof of A without A even being a logical consequence of $A_1, \ldots, A_n$. A proponent of this view might argue that in proofs in physics the relevant mathematics is part of the context and need not be stated. (This last view was suggested by Paul Benacerraf.)

We leave these issues aside since they concern all formal analyses of inference, whether first order or second order.

CHAPTER II

In this chapter we describe second order logic and establish some simple relationships between second and first order languages. We show that given an interpreted language L of one sort, there is an interpreted language L' of the other and a one-one translation which preserves truth conditions. In one direction this conclusion is immediate since first order logic is a sublanguage of second order logic. The other direction results from our ability to express any statement about the subsets of a set in a first order language possessing a symbol for the membership relation and whose domain contains these subsets. We also show that given a theory T couched in L, there is a T' couched in L' such that a formula is provable in T if and only if its translation is provable in T'. Given axioms and rules of inference for L, we simply take their translations as the axioms and rules for L'. First and second order languages and theories thus do not differ on which statements are expressible or provable.

First and second order logic differ in their conceptions of logical form, as expressed in their definitions of truth under an interpretation. A second order sentence and its first order translation have the same truth conditions, but have a different logical form and differ in their place in the consequence relation. The differences between first and second order logic stem from this difference in logical form.

We begin by describing second order logic (L2). Add denumerably many one-place predicate variables $P_1, P_2, \ldots$ to some standard system of first order logic with identity.[1] Alter the formation rules so that each P_i can occur in place of one-place predicate letters and be bound by quantifiers. If $x_1 . x_2, \ldots$ are the individual variables of first order logic, then we now have formulas like $(\exists P)Px$.[2]

An interpretation M of L2 is just an interpretation of its first order sublanguage. That is, M is an ordered pair $\langle D,F \rangle$, where D is the domain and F is a function mapping each constant, n-place function

symbol and n-place relation symbol to an element of D, an n-place function from D to D or an n-place relation over D, respectively. We define truth under an interpretation M recursively, using evaluations of the variables of L2. An evaluation i is a function mapping the individual and predicate variables of L2 to elements and subsets of D, respectively. The definition uses the standard notion of a formula being satisfied by an evaluation. The clauses for the connectives and for quantification with respect to an individual variable are the same as for the first order languages. The clause for formulas of the form $(\exists P)\ A(P)$ is

> $(\exists P)\ A(P)$ is satisfied by $M = \langle D,F \rangle$ relative to evaluation i if and only if there is an evaluation i' such that A(P) is satisfied by M relative to i', where i and i' differ at most in that $i(P) \neq i'(P)$.

A sentence of the form $(\exists P)\ A(P)$ thus asserts that there exists a subset of the domain satisfying A(P) when assigned to P. The rest of the definition is identical to that for first order languages.

We want to compare the expressive power of interpreted languages based on L2 with those based on first order logic. Is a statement expressible in a second order language if and only if it is expressible in a first order language? Two sentences express the same statement if they have the same truth conditions, and we can identify the truth conditions of a sentence with its truth definition. Two sentences have the same truth conditions if their respective truth definitions are identical or can be made so by substitution of co-referential terms.

Since first order logic is a sublanguage of L2, one direction is immediate. An interpretation of a first order language is also an interpretation of the corresponding second order language possessing the same nonlogical vocabulary. Since the definition of truth under an interpretation for second order languages extends that for first order languages, any statement expressible in a first order language is expressed by the same sentence in the corresponding second order language.

The other direction takes longer to show, but it is also straightforward. Suppose we have made a particular choice of constants, function symbols, relation symbols and an interpretation for L2, giving us an interpreted second order language. We will show that there is an interpreted (two-sorted)[3] first order language L1 and a translation h mapping the formulas of L2 one-one into those of L1 such that A and h(A) have the same truth conditions. The logical symbols of L1 are the usual connectives and quantifiers, = (identity), sort symbols I (for individuals) and C (for classes), denumerably many variables $x_1, x_2, \ldots$ of sort I and denumerably many variables $P_1, P_2, \ldots$ of sort C. The nonlogical symbols of L1 are those of L2 with the addition of a two-place relation symbol E. Each constant is of sort I. A <u>type</u> is a finite sequence $(S_1, \ldots, S_k)$ of sort symbols, and each n-place function and relation symbol has a unique type of length n+1 and n, respectively. In L1 the type of each n-place function symbol is the n+1-tuple (I, . . .,I). The type of = is (I,I), that of E is (I,C) and that of any other n-place relation symbol is the n-tuple (I, . . .,I). The formation rules are the usual ones, with the addition of type restrictions. If f is a function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term--of sort I--if and only if each t_i is of sort I. If R is a relation symbol of type $(S_1, \ldots, S_n)$ and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is a formula if and only if each t_i is of sort S_i.

An interpretation G of L1 is a function whose domain consists of the sort symbols and the nonlogical symbols of L1 and such that

(1) for each constant c of sort S, G(c) is an element of G(S),

(2) for each n-place function symbol f of type $(S_1, \ldots, S_n, S_{n+1})$, G(f) maps $G(S_1) \times \ldots \times G(S_n)$ into $G(S_{n+1})$,

and

(3) for each n-place relation symbol R of type $(S_1, \ldots, S_n)$, G(R) is a subset of $G(S_1) \times \ldots \times G(S_n)$.

The definition of truth under an intepretation G employs evaluations mapping the variables of sort S to elements of G(S), and the notion of the

the satisfaction of a formula relative to an evaluation. The clauses for the connectives are the usual ones. The clause for the existential quantification of a variable of sort S is

> $(\exists z)$ A(z) is satisfied by G relative to evaluation g if and only if there is an evaluation g' such that A(z) is satisfied by G relative to g', where g and g' differ at most in that $g(z) \neq g'(z)$.

If $M = \langle D,F \rangle$ is the interpretation of L2, we assign L1 the interpretation M* defined as follows. Let $M^*(I) = D$, $M^*(C) = \wp(D)$,[4] $M^*(E) = D \times \wp(D)$ and $M^*(\alpha) = F(\alpha)$ for any other nonlogical symbol α of L1.

The translation h mapping L2 one-one into L1 is the natural one. If $t_1, \ldots, t_n$ are terms and R is an n-place relation symbol, let $h(R(t_1, \ldots, t_n)) = R(t_1, \ldots, t_n)$. For any predicate variable P_1 and individual variable x_j, let $h(P_i x_j) = E(x_j, P_i)$. For complex formulas let $h(A \wedge B) = h(A) \wedge h(B)$, $h(\sim A) = \sim h(A)$ and $h((\exists z)\, A) = (\exists z)\, h(A)$.

Then a formula A of L2 has the same truth conditions under M relative to an evaluation i as its translation h(A) has under M*,i.[5] Since terms and relation symbols receive the same interpretation under M and M*, $R(t_1, \ldots, t_n) = h(R(t_1, \ldots, t_n))$ has the same truth conditions in both languages. For any individual and predicate variables, Px is satisfied by M,i if and only if $i(x) \in i(P)$. Since $M^*(E) = \in \restriction D \times \wp(D)$, $h(Px) = E(x,P)$ has the same truth conditions under M*,i. For complex formulas we need only consider those of the form $(\exists P)$ A. $(\exists P)$ A is satisfied by M,i if and only if there is an evaluation i' such that A is satisfied by M,i' where i and i' differ at most in that $i(P) \neq i'(P)$. Its translation, $(\exists P)\, h(A)$, is satisfied by M*,i if and only if h(A) is satisfied by M*,i' for some i' of the same type. By the induction hypothesis, these two formulas have the same truth conditions.

This equivalence in the expressive powers of second order and first order languagesis not restricted to those--like L2--having only one-place predicate variables. Some second order languages contain, for each natural number n>0, a denumerable number of n-place predicate

variables. For any language $L2_n$ of this type, there is a corresponding first order language in which we can express the same statements. Let $L1_n$ be L1 with the addition of a new n+1-place relation symbol E_{n+1} for each n>1. Each E_{n+1} is of type $(S_1, \ldots, S_{n+1})$, where $S_i = I$ for $1<i<n$ and $S_{n+1} = C$. Given an interpretation of $L2_n$, the corresponding interpretation is as before, with the added requirement that each E_{n+1} is interpreted to be satisfied by $a_1, \ldots, a_n, a_{n+1}$ just in case $\langle a_1, \ldots, a_n \rangle \in a_{n+1}$. If we extend h so that, for an n-place predicate variable P, $h(P(x_1, \ldots, x_n)) = E_{n+1}(x_1, \ldots, x_n, P)$, then any sentence A of $L2_n$ has the same truth conditions under an interpretation of $L2_n$ as h(A) has under the corresponding interpretation of $L1_n$.

Theories couched in first and second order languages are also equivalent with regard to provability. Again, one direction is obvious. Axioms and rules of inference for a first order language are also axioms and rules of inference for (the first order portion of) a second order language. Since the sentences of a first order language have the same truth conditions in the corresponding second order language, any statement provable in a first order language will be provable by the same axioms and rules in a second order language. In the other direction, given a set Ax of axioms and a set RI of rules of inference defining a theory T2 in the language of L2, there is a theory T1 in the language of L1 such that $\vdash_{T2} A \leftrightarrow \vdash_{T1} h(A)$ for any formula A. The axioms of T1 are the translations under h of the axioms of T2. For each rule R of T2, L1 has the rule R' allowing one to infer h(A) from $\{h(B_1): i \in I\}$ whenever A follows from $\{B_i: i \in I\}$. Then a formula A is provable in T2 just in case a formula with the same truth conditions is provable in T1. L1 and L2 do not differ in the assertions provable in theories couched in these languages.

The differences between L1 and L2 concern the logical form of their sentences. A sentence A of L2 and its translation in L1 have the same truth conditions and A is provable in some T2 if and only if h(A) is provable in the corresponding T1, but they do not have the same logical form. We saw in Chapter I that the definition of truth under an interpretation

for a formal language specifies the logical form of its sentences by showing how the truth value of a sentence depends on the meaning of its significant parts. Here then is the difference between L1 and L2. In L2 subset is treated as a logical concept. In every interpretation the predicate variables range over all subsets of the domain D of the individual variables and Px is satisfied by an evaluation i just in case $i(x) \varepsilon i(P)$. We can express the same statements in L1 by letting the variables of sort C range over the powerset of the domain over which the variables of sort I range and taking E to be the restriction of the membership relation. But this is not required by the semantics. Any sets may be chosen for sorts I and C, and E can be any two-place relation between their members.

Their different conceptions of logical form result in different consequence relations, that is, h does not preserve the consequence relation. For example, consider any sentence of L2 of the form (1) $(\exists P)(\forall x)(Px \leftrightarrow Bx)$, where Bx is any open formula with one free variable and P does not occur in Bx. There is a subset of the domain of any interpretation containing those elements of the domain which satisfy Bx when assigned to x. Since in any interpretation of L2 the predicate variables range over all subsets of the domain, (1) is true under every interpretation and thus is valid. Given the first order semantics of L1, however, there are interpretations of L1 under which h(1), that is, $(\exists P)(\forall x)(E(x,P) \leftrightarrow h(B))$, is false. It will be false, for example, under an interpretation G in which $G(C) \subseteq \wp(G(I))$, $G(E) = \in \upharpoonright G(I) \times G(C)$ and where the subset of G(I) defined by h(B) is not a member of G(C). Similarly, L2 and first order languages differ on properties whose definition requires quantification over all interpretations. L1 is complete and compact; L2 is neither.[6]

The distinction between L2 and L1 cannot be drawn on syntactic grounds. One might think there is an obvious syntactic difference between such languages since only in L2 do quantifiable variables appear in predicate position. But to distinguish first and second order languages, 'quantification into predicate positions' must be understood

semantically. L1 can be rewritten as L1' with primitive formulas of the form Px instead of the form E(x,P), while leaving the semantics essentially unchanged. An interpretation of L1' is a function G' whose domain contains the sort symbols and nonlogical symbols of L1' and an arbitrary object, say 0. G' provides an interpretation, subject to type restrictions, for the sort symbols and nonlogical symbols of L1', and G'(0) must be a subset of G'(I) X G'(C). The definition of truth under an interpretation is that for L1, with the change that Px is satisfied by G',g if and only if $\langle g(x),g(P)\rangle \in$ g'(0). To each interpretation G of L1 there corresponds the interpretation G* of L1' in which G*(α) = G(α), when α is a sort symbol or any nonlogical symbol common to L1 and L1', and G*(0) = G(E). Then each formula of L1 has the same truth conditions under G,g as its translation in L1' has under G*,g. Since this correspondence between interpretations of L1 and those of L1' is one-one and onto, the translation from L1 to L1' preserves logical form and logical consequence. L1' is a syntactic variant of L1 and is still a first order language. Similarly, we could form a second order language L2' with all variables in subject position.

Since L1 and L2 differ in their logical form, logical theories based on them will give different accounts of logical form and logical consequence in natural language. A second order logical theory will hold some sentences asserting the existence of sets to be valid, while a first order logical theory will not. A theory based on L2 might hold (2) 'There is a set P of natural numbers containing 0' to be valid. If L2 has a constant $\bar{0}$ and is given an interpretation M = $\langle$D,F$\rangle$ such that D is the set of natural numbers and F($\bar{0}$) = 0, then (2) can be formalized by the valid second order sentence ($\exists$P) P$\bar{0}$. (2) can also be formalized by ($\exists$P)E($\bar{0}$,P) = h(($\exists$P)P$\bar{0}$) in L1 under the interpretation M*, but ($\exists$P)E($\bar{0}$,P) is not valid. Both sentences assert that there is a set of natural numbers containing 0 and thus both formalize (2). Yet on the first formalization it is valid, while on the second it is a nonlogical truth. A second order logical theory need not hold that (2) is valid,

for (2) can also be formalized by a nonlogical truth in the first order sublanguage of L2. But any second order account of logical form will take some sentences employing quantification over sets to be valid which a first order account construes as nonlogical truths. The controversy here is whether a logical theory based on L2 can provide a correct account of logical form and logical consequence in natural language.[7]

NOTES

1. For example, take the formulation in Elliott Mendelson's <u>Introduction to Mathematical Logic</u> (Princeton, 1964).

2. We omit subscripts when this will cause no confusion.

3. The use of a two-sorted first order language is not essential since any statement expressible in such a language is expressible in a one-sorted first order language with one-place predicates serving the function of the sorts.

4. $\mathcal{P}(x)$ is the powerset of x.

5. Since $M^*(I) = D$ and $M^*(C) = \mathcal{P}(D)$, the evaluations of L2 and L1 are identical.

6. A formal language is <u>complete</u> if there exists a recursive proof procedure P such that A is a logical consequence of X if and only if A is provable from X by means of P. A formal language is <u>compact</u> if a set X of sentences has a model if and only if every finite subset of X has a model. (A <u>model</u> of a set of sentences is an interpretation under which every member of the set is true.)

7. Since both first and second order languages are extensional, neither a first order nor a second order account of logical form can provide the logical form of sentences containing intensional idioms. Accordingly, when comparing these logics we can restrict our attention to the extensional portion of natural language.

CHAPTER III

A common feeling is that second order logic--whatever its merits--is not as natural as first order logic. In this connection it is salutary to remember that Frege's logical systems were second order. The birth of modern logic is usually placed in 1879 when Frege published Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought.[1] In this short booklet Frege presented a formal system of the second order predicate calculus. Frege saw no special significance in the first order part of his system. Indeed, none of his philosophical views would have precluded his using third or fourth order quantification if the need had arisen, e.g., if the analysis of some concept from arithmetic had required it.

In this chapter we consider Frege's reasons for employing second order quantification in his logical systems. First the role his formal systems played in his logicist program is described. After discussing some related issues, we reconstruct Frege's justification for his second order systems. We then conclude with a brief discussion of the view that Frege reduced arithmetic to set theory.

Frege's great achievement was to develop a formal system in which one could exhibit the logical form of sentences involving multiple quantification, like (1) 'Everything is bigger than something.'[2] Before Frege, though systems of logic might deal with varying success with the propositional connectives or provide a sophisticated system of class inclusion, they foundered on the problem of multiple quantification. Frege was the first logician to succeed in developing a formalism in which one could represent the logical form of such sentences, e.g., in which one could distinguish (2) $(\forall x)(\exists y)$(x is bigger than y) and (3) $(\exists y)(\forall x)$(x is bigger than y) as alternate readings of (1).

Frege says in the preface to Begriffsschrift that he was led to this discovery through his interest in the foundations of arithmetic. He wondered whether the laws of arithmetic rest solely on laws of pure

logic--on those laws 'upon which all knowledge rests' and, accordingly, which contain no reference to 'the particular characteristics of objects.'[3] He was not interested in how anyone actually came to know the laws of arithmetic, but in how those laws could be given their most secure foundation. Here Frege did not only want to exclude, e.g., the case in which we know that 1+1 = 2 because we accept it on authority from our first grade teacher. We may believe this proposition because we can derive it from the Peano axioms, which we find self-evident. We might even provide the proof in a formal system. Even so, Frege would still think it is an open question whether this proof shows the ultimate grounds on which this proposition rests. He believed that if we could show that 1+1 = 2 followed from laws that were more basic than the Peano axioms, then any derivation from these axioms would not constitute the <u>real</u> proof of the proposition, for it would not display the ultimate grounds on which its truth rests.[4]

Frege attempted to provide the epistemological foundation of the truths of arithmetic by showing that they follow from the laws of logic. But he soon encountered an obstacle--natural language. In attempting to prove some arithmetic truth from laws of logic, how could he be certain that he had not tacitly employed any unacknowledged assumptions, e.g., which went unnoticed due to their obviousness? Without this assurance he could not be certain that the proof didn't require some nonlogical assumption. To solve this problem Frege invented a formal language adequate to express the propositions of mathematics. He then provided effective, syntactically specified rules of inference and required that each sentence in a proof either be listed as an assumption or else result from the application of some rule of inference to earlier lines. Frege claimed that his rules of inference could be seen to yield only logical consequences of sentences to which they were applied. Since these rules were finite in number and effective, he now had a mechanical means of checking putative proofs. If the laws of arithmetic could be derived by means of these rules of inference

from logical laws, then Frege could be certain they followed from the laws of logic alone.

To create a formal language in which one could express mathematical propositions, Frege needed a new analysis of the structure of natural language. Sentences that were not constructed using the propositional connectives were traditionally seen as composed out of a subject and a predicate. The subject term was held to denote an object and the predicate term to denote or express a property. This analysis, however, was unable to account for the logical relationships between sentences containing either multiple quantification or relation terms. There were also problems concerning the denotation of subject terms like 'something' and 'everything.'

Frege declared the subject-predicate distinction to be irrelevant to a logical analysis of language. Finding a model in the use of functions in mathematics, Frege analyzed sentences into the application of (one or more) functional expressions to (one or more) argument expressions. When he wrote Begriffsschrift in 1879, Frege used this model intuitively, without possessing a coherent semantic theory. But in 1891-3 in 'Function and Concept,'[5] 'On Sense and Reference'[6] and the first volume of The Basic Laws of Arithmetic, Frege developed a sophisticated semantic theory based on this idea. In arithmetic we see 5 as the value of the function x+y for the arguments 2 and 3. On the linguistic level, we see '2+3' as a name denoting 5 which is constructed out of the proper names '2' and '3' and the functional expression 'x+y'.[7] Frege seized on this model for the analysis of certain mathematical expressions and, like a good mathematician, generalized it. In the first place, he held that a nonmathematical name like 'the capital of France' could also be seen as resulting from the completion of the functional expression 'the capital of x' by the proper name 'France.' The result is a name denoting an object, Paris, which is the value of the function denoted by 'the capital of x' when applied to France as argument.

Frege went on to analyze sentences as formed by the completion of a functional expression by the name of an argument. 'John is bigger than Mary' could be analyzed as the result of completing the function name 'x is bigger than y' by the names 'John', 'Mary' (in that order).[8] Here Frege held that the value of the function named by 'x is bigger than y' for the arguments John, Mary is a truth value. Functions that always have a truth value as value for any argument are called concepts. If a concept maps an object or pair of objects to the True, then that object or pair is said to fall under that concept.

In addition to first level functions which map (ordered sequences of) objects to other objects, Frege held that there are second level functions mapping first level functions on to objects. The most important examples of such functions are the first order existential and universal quantifiers. 'John is bigger than something' results from completing the second level functional expression $(\exists x)A(x)$ by the first level functional expression 'John is bigger than x'. The corresponding second level function maps a first level function to the True just in case there exists some object which the first level function maps to the True.

Frege went on to construe quantification over functions or properties as a third level function taking second level functions as arguments and having truth values as values. For example, 'There is a property F and an object x such that Fx' results by completing the third level functional expression 'There is an F such that A(Fx)' by the second level functional expression 'There is an x such that Gx.'[9] This third level functional expression denotes a third level function whose value for any second level function as argument is the True just in case there exists a first level function which the second level function maps to the True--that is, for our example, just in case there exists a first level function and an object which this function maps to the True.

This semantics built upon the model of the application of function to argument allowed Frege to provide an account of the truth conditions of sentences couched in extensional idioms. For any complex sentence of this sort Frege's semantics demonstrated how its denotation, that is, its truth value, depended upon the denotations of its significant parts. Using this analysis, Frege created a formal language in which any sentence employing only extensional idioms could be expressed once the appropriate simple proper names and simple functional expressions had been added. Since in such a formal language the bewildering variety of forms of natural language is replaced by a limited and precisely specified number of syntactic operations whereby complex sentences are formed, the task of specifying effective, syntactically defined rules of inference is manageable.

In Begriffsschrift Frege did not have a coherent semantic theory. There is a striking contrast in this early work between the sophistication of his formal system and the primitive nature of his semantics. His formal system is a formalization of the second order predicate calculus, containing as a subsystem a complete formalization of the first order predicate calculus. But his reflections on semantics are crude in comparison with his later views, and at times they are simply confused.

When writing Begriffsschrift Frege had not yet made a distinction between the sense and the reference of an expression. Instead he simply speaks of the 'conceptual content' of an expression, explaining that this is that part of its meaning which is relevant to determining the logical consequences of the expression. He says that he will analyze sentences into function and argument, rather than subject and predicate, but says little about the nature of functions. Functions are never even described as mapping arguments on to the value of the function for those arguments. Restricting his attention to those contents which can become possible contents of a judgment, he merely says that we obtain a proposition when a function is completed by an argument.

Frege illustrates how there are often various ways to analyze an expression into a functional expression and one or more argument expressions. Rather than extend this analysis to the content of an expression, however, he explicitly defines functions as parts of an expression seen as constant and arguments as parts seen as replaceable by other expressions.[10] Yet soon after, he implicitly abandons this account by explaining $(\forall x)\ A(x)$ as 'for every argument x, A(x) is a fact.' Given his account of functions and arguments, this should mean that for every expression x as argument, A(x) is a fact. Yet throughout Begriffsschrift Frege interprets quantification objectually. For example, he reads $\sim(\forall x)\ A(x)$ as 'there are some objects that do not have the property A.'[11]

In Begriffsschrift Frege barely had the semblance of a theory of language and what he had was confused. What Frege did have, in the application of function to argument, was a model for the analysis of language, however imperfect was his theoretical understanding of that model. Using this model, in this first work Frege presents a formal system of the second order predicate calculus with identity. He takes the material conditional, negation and the universal quantifier as primitive, indicating how the existential quantifier and the other propositional connectives can be defined. Frege says that we may regard a proposition Pc either as a function of the argument c or else as a function of the argument P. We can thus not only form the judgment, $(\forall x)\ P(x)$, but also the judgment $(\forall f) f(a)$. The first is the judgment that, whatever object a we take as argument, Pa obtains. In the second judgment we see Pa as a function of the argument P. This judgment is the judgment that, whatever function f we take as argument, f(a) is a fact.[12]

In Begriffsschrift there are nine axioms schemas, and modus ponens is the only rule of inference. Using his account of the truth conditions of sentences of his system, Frege argues that each instance of these schemas is valid and that modus ponens is sound. In his discussion of the use of variables, however, Frege incorporates two further rules: the rule of generalization and the rule allowing one to infer $A \rightarrow (\forall x)\ B$

from $A \rightarrow B$ provided that x does not occur free in A.[13] These transformations are again justified by reference to the truth conditions of sentences. Frege requires that in a proof each step either be an axiom or else be obtainable from earlier lines by means of a rule of inference. These axioms and rules provide a complete formalization of the first order portion of his system.

The formal system in Begriffsschrift removed the obstacle Frege had encountered in attempting to determine if the truths of arithmetic were derivable from the laws of logic. The rules of inference of Begriffsschrift could be seen to be sound and effective. The soundness of the rules assured Frege that any provable proposition is a logical consequence of the axioms. Their effectiveness allowed him to obtain the certainty he desired, for confronted with an argument he had a purely mechanical means of determining, in a finite number of steps, whether or not it was a proof. If a check revealed that the argument was a proof in his system, then he knew that the conclusion followed logically from the axioms and that no extra assumptions had been tacitly employed. Since he was confident that his axioms were laws of logic, Frege felt justified in maintaining that a proof in his system showed that the proposition proved followed logically from the laws of logic.

With this obstacle removed, Frege could proceed with his reduction of (cardinal) arithmetic to logic, in which second order quantification plays a crucial role in several key definitions. Frege first defines 'there are the same number of F's as G's,' which we can abbreviate as NxFx = NxGx.[14] Using the same idea Cantor employed, he defines

$$NxFx = NxGx \overset{\text{df.}}{\leftrightarrow} (\exists H)\ [\text{H is a one-one correspondence between the F's and the G's}],$$

where H is a second order variable.[15] He goes on to define NxFx as 'the class of concepts G such that G is numerically equivalent to F'[16] and 0 as $Nx(x \neq x)$. Second order quantification is used again in Frege's definition of successor:

$$y \text{ is the successor of } x \overset{df}{\leftrightarrow} (\exists P)(y = NzPz \wedge (\exists w)(Pw \wedge Nz(Pz \wedge z \neq w) = x)).$$

He then shows that, as defined, 0 and the successor relation have the usual properties. Finally, Frege uses second order quantification to define the natural numbers. In Begriffsschrift 'y is a member of the R sequence beginning with x' is defined as

$$y = x \vee (\forall P) \ [(\forall w)(\forall z)(((Pw \wedge R(w,z)) \rightarrow Pz) \wedge (\forall u)(R(x,u) \rightarrow Pu)) \rightarrow Py].$$ [17]

That is, y is a member of the R sequence beginning with x if and only if either y=x or, for any property P, if P is closed under R and also closed under applications of R to x, then y has property P. Using the successor relation S(x,y) as defined above, Frege defines 'x is a natural number' as 'x is a member of the S series beginning with 0', i.e.,

$$x = 0 \vee (\forall P)[(\forall w)(\forall z)(((Pw \wedge S(w,z) \rightarrow Pz) \wedge (\forall u)(S(0,u) \rightarrow Pu)) \rightarrow Px].$$

From this second order definition it is easy to show that complete induction holds for the natural numbers. Second order induction for the natural numbers can be expressed as

$$(\forall P)[(P0 \wedge (\forall w)(\forall z)((Pw \wedge S(w,z)) \rightarrow Pz)) \rightarrow (\forall x)(x \text{ is a natural number} \rightarrow Px)],$$

and this can be seen to follow immediately from Frege's definition of 'x is a natural number.'

We have said that Frege's system in Begriffsschrift is second order and that he provided an account of the logical form of sentences of natural language by formalizing them in this system. These claims are not so straightforward, however. Unlike the predicate variables in L2, those in Frege's system do not range over sets. Frege held that they range over concepts and that sets, being objects, are distinct from concepts. Also, in Chapter I we said that the logical form of the sentence of a formal language is provided by the definitions of an interpretation and of truth under an interpretation. Yet although Frege provides an informal specification of the truth conditions of sentences, he never uses the idea of an interpretation of a formal language.

What we do is consider a formal language for which we have a notion of interpretation and of truth under an interpretation and which provides the most faithful modern formulation of Frege's intentions. There are several reasons why such a language will be second order. First, although Frege held that concepts were never identical with sets, on his conception of concepts there is exactly one concept true of just the objects in any arbitrary set. Each set is thus associated with a unique concept. Second, Frege held that names for concepts and variables ranging over concepts can never appear in subject position. A sentence like $(\exists P)(\exists Q)$ $(P=Q)$ is not permissible. This is preserved in standard formulations of second order logic. Finally, sentences containing predicate variables which Frege held to be logical truths, like $(\forall P)$ $(Pa \rightarrow Ga)$,[18] are also logical truths in a modern formulation of second order logic.

Frege created his formal language so that he could have an effective way of checking the validity of putative proofs. If an argument is couched in his symbolism and proceeds by the application of his rules of inference, then he could be certain that the conclusion is a logical consequence of the premisses. If the argument is stated in natural language, then to determine if the conclusion A is a logical consequence of the set of premisses X we must search in Frege's system for a formal proof of A', the formalization of A, from X', the set of formalizations of members of X. Since Frege's rules of inference are finite in number and effective, we can recognize a proof of A' from X' if we find one; and if we do, then we know that A follows logically from X. But Frege's system does not provide an effective check for the validity of arguments stated in natural language. The formalization of sentences of natural language is not an effective operation. The choice of a correct formalization requires ingenuity in reducing the various constructions in some natural language sentence to the limited number available in some formal language. The choice of a correct formalization of certain types of sentences may not only not be mechanical, but

may pose philosophical problems. An example is the dispute over the correct formalization of action sentences, such as 'Jones listened carefully to the instructions.'[19]

If we confine ourselves to the formal language in Begriffsschrift, then we can simply check whether a putative proof proceeds by application of the rules of inference. If it doesn't, of course, the conclusion may still be provable from the premisses. Yet there is no effective means to determine whether a sentence in Frege's system is provable from some set of premisses. All complete formalizations of the first order predicate calculus are undecidable, that is, there is no mechanical procedure by which to determine in a finite number of steps whether or not a given sentence is provable. Since Frege's system contains a complete formalization of the first order predicate calculus, it is undecidable as well. Further, if some formal sentence B is not provable from some set Y, this does not show that B is not a logical consequence of Y since Frege's system is not complete. We now know that there does not exist a recursive proof procedure for second order consequence which is both complete and consistent. Although Frege never discussed questions of completeness, he seemed to believe that his systems were complete. This belief no doubt resulted from his practical success in finding a proof of B from Y whenever he had good reason to believe that B was a logical consequence of Y.

But why did Frege think that second order quantification was part of logic? Today we are accustomed to thinking of the first order predicate calculus as a 'natural' system. It is not hard to feel that any departure from first order logic, at least in the formalization of extensional idioms, requires special justification. Yet though Frege's formal systems were second order, he never felt that the use of second order quantification required any special justification or was in any way less natural than first order quantification.

Now it is not surprising that Frege should have been unaware of the philosophical problems surrounding a second order logic. After all, even to state these problems we must employ concepts and mathematical results which were not available to Frege. Yet unless Frege had good reason to believe that second order quantification was part of logic, he would have no justification for the claim that he had reduced arithmetic to logic.

Actually Frege did possess a justification for the use of second order quantification in his system of logic, though this justification was stronger at the time he wrote The Basic Laws of Arithmetic than it was when he wrote Begriffsschrift. In The Basic Laws of Arithmetic Frege possessed a well confirmed semantic theory justifying second order quantification. In Begriffsschrift, on the other hand, there is no general semantic theory nor even a coherent account of the nature of a function. All Frege had, in the application of function to argument, was a model for the analysis of language, and one could have employed this model without allowing quantification over functions.

Yet the very success of Begriffsschrift provided justification for the use of second order quantification. Using second order quantification, Frege had created a formal language in which he could represent any proposition of mathematics, including those involving relations and multiple quantification. Using this notation he could provide an acceptable account of the truth conditions of these sentences. We must remember that at this time there was no other logical theory which could stand comparison with Frege's. There were many systems of class inclusion, but there was no other logical theory which could provide a general account of quantification. Today we know that the truth conditions of quantified sentences can also be explained by formalizing them in a first order language. Any advocate of second order logic must provide good reasons why it should be preferred to first order logic. Frege, on the other hand, had a strong justification for his second order system since it was the only logical system that gave intuitively acceptable solutions to problems that had exercised logicians for centuries.

By the time the first volume of The Basic Laws of Arithmetic appeared in 1893, Frege had an even stronger justification for the use of second order quantification. For by 1893 Frege could justify the use of second order quantification in his logical systems by reference to a sophisticated semantic theory. Further, this theory had strong support.

Let us begin with the first point. On Frege's semantic theory each significant part of an expression is held to denote some entity. The denotation of a complex expression is explained as the result of the application of a function to one or more arguments. To take a simple case, in an atomic expression of the form R(a,b), the proper names a and b denote objects and the relational expression R(x,y) denotes a relation. The truth value, or denotation, of R(a,b) is the value of the relation denoted by R(x,y) for the objects denoted by a and b as arguments (in that order). Since on this theory the proper name a denotes an object, there can be no objection to quantifying over the position it occupies to arrive at $(\exists x)\ R(x,b)$ by existential generalization. But given this semantic theory, the move to $(\exists S)S(a,b)$ is equally justified. Just as a refers to an object, R refers to a relation. If t is a name occurring in a sentence A(t), surely we can form the sentence $(\exists k)\ A(k)$ which says that there exists an entity which has the same properties that A(t) asserts to hold of the denotation of t. Once we hold that functional expressions are denoting names, one would need a special justification for prohibiting quantification over positions occupied by these expressions.[20]

In addition, Frege's semantic theory was well supported. The most important source of this support came from the success of Frege's formal systems. In Begriffsschrift an account of the truth conditions of sentences is given which employs an intuitive notion of the application of function to argument and an explanation of the meaning of the connectives and quantifiers. We saw that Frege did not have a coherent theory of language until 1891-3. Then he could defend his earlier

analysis of sentences into connectives, quantifiers and the application of function to argument by embedding it in a coherent semantic theory. This theory explained connectives and quantifiers as functions of a certain kind and provided a general account of the nature of functions, the distinction between functions and arguments and the distinction between functions and objects. The intuitive model which Frege used in *Begriffsschrift* was now replaced by an explicit theory. Since this theory justified Frege's formal systems of quantification, the success of these systems in turn provided support for this semantic theory.

An attractive feature of Frege's theory was its great generality and simplicity. It not only gave an account of the semantics of extensional contexts, but also made a start at an account of intensional contexts. Using the notion of sense, Frege's theory provided an explanation why two sentences may differ only in two extensionally equivalent parts and yet convey different information. All this was achieved by a wonderfully simple theory. The only basic concepts are sense, reference and the application of function to argument. From this meager beginning Frege provided a plausible, coherent semantic theory without *ad hoc* features.

Finally, this semantic theory received support by being the only theory at the time which had a chance of success. Just for a start, no other theory could explain the truth conditions of quantified sentences. Only Frege's theory even had a chance of providing a correct account, e.g., of the truth conditions of the sentences of mathematics. So by 1893 Frege could defend second order quantification, not only by its practical success in *Begriffsschrift*, but also by reference to the only simple, well confirmed semantic theory in existence. Though Frege never attempted to justify the use of second order quantification in his logical systems, he had a strong justification available. Today this justification is no longer compelling. We now have first order formalizations of mathematical theories and have alternative semantic theories which do not attribute reference to predicates and functional expressions. In

addition, we are aware of metamathematical results concerning the completeness, compactness and categoricity of (theories couched in) first and second order languages which bear on the matter. Frege is understandably silent on these issues around which the debate over second order logic now centers.

We conclude this chapter with a brief discussion of the common view that Frege reduced arithmetic to set theory in the belief that set theory was logic. There is some truth in this claim--after all, Russell's Paradox is derivable in the formal system of The Basic Laws of Arithmetic. Yet this view also obscures much that Frege considered essential and can only be affirmed with qualification. We will indicate how it must be qualified so that we can reconstruct the way Frege saw these issues.

In discussions of the logicists, second order logic is commonly treated as part of set theory. This view has recently been criticized by George Boolos in 'On Second-Order Logic.' The essential difference between set theory and second order logic is that second order logic is not an interpreted theory in which we quantify over all sets. It is an uninterpreted formal language which can be interpreted in various ways. Set theory, on the other hand, is an interpreted theory in which we quantify over all sets and formulate assertions about sets using the membership relation.

Even if we refuse to identify second order logic with set theory, we must admit that The Basic Laws of Arithmetic did contain a system of set theory. Frege held that the extension, or course-of-values, of any function is an object. Since Frege held that concepts are functions, the course-of-values of, e.g., a first-level concept $A(x)$ is the set of ordered pairs $\langle x,y\rangle$ such that x is an object and y is the True if x falls under A and is the False otherwise. The course-of-values of the function $A(x)$ is $'\varepsilon(A(\varepsilon))$. Although he had his doubts,[21] Frege laid down the infamous Basic Law V which asserts that two courses-of-values are identical just in case they are courses-of-values of functions which

are extensionally equivalent, that is,

$$[\text{'}\varepsilon f(e) = \text{'}\alpha g(\alpha)] = [(\forall x)(f(x) = g(x))].$$ [22]

In this system we can certainly prove that many sets exist--too many as Frege eventually learned. To prove that some extension exists, we must produce a function f(x) whose extension it is; then 'εf(ε) will denote the extension in question. This extension is an object and thus an admissible argument to every first-level function, including f(x) itself if f(x) is a first-level function. In this way we can generate Russell's Paradox. Further, in Frege's theory cardinal numbers are identified with certain extensions. These are the main reasons for holding that Frege reduced arithmetic to an (inconsistent) formalization of set theory.

Though this picture is essentially correct, there is a major qualification which must be kept in mind if we are to appreciate Frege's view of these issues. For Frege believed that the use of concepts is essential in formulating a theory of sets. He held that concepts are epistemologically prior to sets since the only way we can distinguish a set of objects from a mere aggregate is by bringing the objects under a concept.[23] What is the difference between the set of states of the United States and the set of counties of the United States? They each constitute the same physical aggregate. When explaining the concept of set, one usually just says that they differ since they have different members--the first set has the states as members while the second has the counties as members. Frege's view is that we can only understand this difference between the two sets by realizing that each set is determined by a concept providing the condition for membership in the set. The first set is determined by the concept <u>state of the United States</u>; the second by the concept <u>county of the United States</u>. Since it is not the case that $(\forall x)$ [x is a state of the United States $\leftrightarrow$ x is a county of the United States], the two sets have different members and thus are distinct. Their associated concepts provide the criterion of

individuation by which we can divide the physical aggregate of the United States into the distinct sets of states and counties.

For this reason Frege would argue that a first order system of set theory cannot provide real proofs of propositions concerning sets. Nor can it provide proofs of arithmetic propositions if we attempt to reduce arithmetic to it. Frege would argue this even if the axioms of the theory were known to be true and the rules of inference were known to be sound. Frege held that a proof of a proposition provides the proposition with its ultimate justification, by showing the most basic truths upon which it rests. In this way a proof reveals the epistemological status of the proposition proved. Since truths about concepts are prior to truths about sets, to prove a proposition concerning sets by appeal to the axioms of some set theory is not to provide a *real* proof, for it does not exhibit the ultimate grounds on which the proposition rests. Frege would argue that the correct justification, and hence the real proof, of propositions concerning sets must start with truths about concepts which have these sets as their extensions. Perhaps a fair summary would be to say that Frege reduced arithmetic to a set theory in which the proposition that some set exists can only be justified by showing that there exists a concept with this set as its extension; and that Frege would have objected on philosophical grounds to any set theory not founded upon concepts in this way.

NOTES

1. In *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*, ed. Jean van Heijenoort (Cambridge, 1971), pp. 1-82.

2. This requires some qualification. For while the first order portion of Frege's system is widely accepted, those who object to higher order logic would claim that no sentence of natural language is correctly formalized by a sentence of *Begriffsschrift* containing a second order quantifier. Also, since Frege didn't employ the idea of an interpretation of his system, there is some difficulty in holding that he provided the logical form of *any* sentences. The latter point will be discussed shortly.

3. Begriffsschrift, p. 5.

4. Since Frege held the Peano axioms to be logical truths, he evidently believed that some laws of logic are more basic than others.

Frege also takes this absolute view of proof in The Foundations of Arithmetic, trans. J.L. Austin (Evanston, 1968), pp. 2-4. This is shown by his speaking of the proof of a proposition. In The Basic Laws of Arithmetic, trans. and ed. Montgomery Frege (Berkeley and Los Angeles, 1967) this view is less apparent. For example, Furth says that there are often several formal proofs of a proposition and implies that some may proceed from different premisses. He still holds, however, that a proof exhibits the epistemological status of the theorem proved and goes on to speak of having shown the grounds upon which each theorem rests. See Basic Laws, p. 3.

5. In Translations From the Philosophical Writings of Gottlob Frege, ed. Peter Geach and Max Black (Oxford, 1970), pp. 21-41.

6. Geach and Black, pp. 56-78.

7. According to Frege we could also see this expression as composed by application of the functional expression '2+y' to '3' or by application of 'x+3' to '2.'

Frege held that, in a functional expression like x+y, x and y are not actually part of the expression. The functional expression is the common element to the names '1+1', '2+1', '2+2', '3+2', . . .; that is, the incomplete expression ____ + ____. The letters x and y are placeholders serving to indicate the kind of supplementation required. Frege used distinctive letters for placeholders and variables, but we use the same letters for both purposes.

8. This sentence can also be analyzed in other ways. See note 4.

9. Here A and G are placeholders for a second-level functional expression and a first-level functional expression, respectively.

10. 'If in an expression . . . a simple or a compound sign has one or more occurrences and if we regard that sign as replaceable in all or some of its occurrences by something else (but everywhere by the same thing), then we call the part that remains invariant in the expression a function, and the replaceable part the argument of the function.' Begriffsschrift, p. 22.

11. Begriffsschrift, p. 27. Further, Frege never claims that the substitutional and objectual readings are equivalent. It is clear that he never intended the substitutional reading to which his explanation of the distinction between function and argument gives rise.

12. Frege stipulates that in such a sentence the function variable only ranges over those functions f such that f(a) is a possible content of a judgment. Such sentences therefore contain quantification over concepts.

13. See van Heijenoort's introduction to Begriffsschrift, pp. 2-3. He points out that Frege also employs some unstated rules of substitution.

14. This presentation largely follows that in Charles Parsons's article, 'Mathematics, Foundations of' in The Encyclopedia of Philosophy, ed. Paul Edwards (New York, 1972), pp. 195-6.

15. Here the matrix can be expressed using only first order quantification.

16. This is the definition in The Foundations of Arithmetic, pp. 79-80. That in Basic Laws differs by replacing sets of concepts by sets of extensions of concepts (p. 100).

Although Begriffsschrift contains no set-theoretic assumptions, in his later work Frege believed that the existence of extensions of concepts was a principle of logic.

17. Begriffsschrift, pp. 55, 59 and 69.

18. This is an instance of Basic Law IIb in Basic Laws, p. 105.

19. See Donald Davidson, 'The Logical Form of Action Sentences' in The Logic of Decision and Action, ed. Nicholas Rescher (Pittsburgh, 1966), pp. 81-95.

20. George Boolos in 'On Second-Order Logic,' Journal of Philosophy 72 (1975), pp. 509-11 argues persuasively against Quine's view that second order quantification is only permissible if functional expressions and predicates are names. Our argument merely relies on the converse: if such expressions do denote, then second order quantification is permissible.

21. Basic Laws, pp. 3-4.

22. Since Frege's final view was that sentences denote truth values, he can use the identity sign instead of the biconditional.

23. Basic Laws, pp. 29-37. Cf. also Geach and Black, pp. 103ff.

CHAPTER IV

People sometimes claim that logical theories based on L2 should be rejected because L2 is not complete.[1] We will argue that the completeness of first order logic does give it an advantage over L2, but not for the reasons usually given. The usual arguments against L2 are based on the claim that only in a logical theory based on a complete logic are all instances of the consequence relation knowable. This epistemological difference is taken to support the general claim that we have reason to prefer complete to incomplete logical theories. The ability to recognize all instances of the consequence relation might be argued to be an *a priori* condition on any acceptable logical theory. We might know *a priori* that an incomplete logic is incorrect. A weaker position is that completeness functions like a methodological canon. Here we have no assurance that the correct logical theory is complete, but all things being equal we adopt complete theories over incomplete rivals. If second order logical theories do not have sufficiently important compensations, their incompleteness is a reason to reject them in favor of first order theories.

We consider these views in turn and reject them. The importance of completeness lies elsewhere. L2 is undesirable because we can never recognize a solution to the inference problem if L2 is correct, while any complete proof procedure solves this problem for first order logic. This conclusion does not show that complete logics are always preferable to incomplete logics since it depends on characteristics of L2 which may not be common to all incomplete logics. Further, to turn it into a convincing rejection of L2 would require resolving difficult issues in epistemology and philosophy of science. For this reason we will reject L2 on the basis of the less contentious criticism given in Chapter V. While contentious, however, the argument in this chapter does provide a plausible reason to consider the incompleteness of L2 a liability.

A formal logic L is complete if there is a recursive proof procedure P such that a formula A is a logical consequence of a set X of formulas if and only if A is provable from X by means of P.[2] We also say that P and any logical theory based on such a logic are complete. Since a logical truth is a logical consequence of the empty set, all logical truths are provable by a complete proof procedure.[3]

The philosophical importance of this technical notion is not obvious, but the assumptions justifying a proof theoretic approach to the inference problem link it to traditional epistemological issues.[4] The crucial assumption is that only a recursively enumerable set of sentences can be known. If that is correct and if each axiom in a complete proof procedure P is known to be valid and each rule of inference is known to be sound, then we can recognize each valid formula by means of a proof in P.[5] On the other hand, if L is incomplete then there are unknowable logical truths. The set of logical truths of a logic L is recursively enumerable if and only if L is complete. If the set of knowable truths is recursively enumerable, then in an incomplete logic some logical truths cannot be known. Since L2 is incomplete, if a second order account of logical form is correct then there exist unknowable logical truths.[6]

On one view this consequence of second order accounts of logical form provides a conclusive reason to reject them. This view holds that an examination of our concept of logical consequence shows that any instance of this relation is recognizable as such. We know *a priori* that any correct account of logical form and logical consequence must be complete. Any relation with unrecognizable instances is not the consequence relation.

Providing a detailed argument against this view is difficult since there is nothing in our informal concept of logical consequence to suggest that each instance of this relation can be known. In Chapter I we characterized the logical form of a sentence as its semantically significant structure. We said that A is a logical consequence of X =

$\{x_i: i\varepsilon I\}$ if the logical form of A and the x_i ensures that A will be true if each x_i is. We then characterized the inference relation using the concept of potential knowledge. A can be inferred from X if there is an argument by which we can come to know that A is a logical consequence of X. The inference relation consists of the knowable instances of the consequence relation. It is clearly included in the consequence relation, but why should we think the converse inclusion holds? Logical consequence was characterized without mentioning either arguments or knowledge, and this characterization provides no reason to believe that whenever A is a logical consequence of X there is a valid argument demonstrating that A follows from X.

We can illustrate this by the following consideration. The logical consequence relation is compact if A is a logical consequence of X if and only if A is a logical consequence of a finite subset of X. If any correct formalization of logical consequence is complete, then the consequence relation must be compact. For if a correct account of logical consequence is complete, then whenever A is a consequence of X then A is provable from X by a complete proof procedure. Since proofs have only finitely many premisses, A is provable from a finite subset of X. But in lieu of possessing a well confirmed logical theory, we have no reason to believe that no instances of the consequence relation depend essentially on the presence of infinitely many sentences. Our informal characterizations certainly don't imply such a restriction on accounts of logical form and logical consequence. We now know that some formal logics, like L2, have consequence relations which are not compact. Whether such a logic can provide a model for the consequence relation in natural language cannot be decided by scrutinizing our informal concept of logical consequence.

If the view that we know _a priori_ that logical consequence and inference are coextensive has so little merit, why has it been so common throughout the history of philosophy? One reason is that the importance of this assumption was not appreciated. Prior to the twentieth century

the epistemological assumption that we can know all instances of the consequence relation seemed irrelevant to the choice of a logical theory--a puzzle for philosophy which logic could safely ignore. Work in the twentieth century on the syntax and semantics of formal languages showed this position is mistaken. If we assume that consequence and inference are coextensive and that only a recursively enumerable set of sentences can be known, then we eliminate prima facie plausible characterizations of logical form, e.g., those provided by second order logical theories.

Yet even before the study of formal languages, the importance of this assumption might have been realized. We mentioned that the assumption that logical consequence and inference are coextensive implies that logical conseuqence is compact. In principle one could have seen that this puts substantive constraints on an acceptable theory, for one could at least have imagined that there should be a plausible account of logical form on which the consequence relation is not compact.

This possibility was never considered, because attention was confined to finite sets of sentences. A typical example is the definition of consequence by Pseudo-Scotus.[7] On this definition a consequence is a proposition with an antecedent and consequent expressing that if the antecedent is true then the consequent must also be true. The idea behind this definition is clear. If we are only concerned with finite sets of sentences, whenever A is a logical consequence of $\{X_1, \ldots, X_n\}$, 'If $X_1 \wedge \ldots \wedge X_n$ then A' is a logical truth. This approach won't work for infinite sets if sentences are of finite length. Since infinite sets of sentences were not considered, the question of the compactness of the consequence relation never arose. Modern developments in set theory and logic changed these habits of thought. Cantor's work in set theory make infinity a topic of mathematical research, and the consideration of infinite sets became common. More important, the invention of formal languages made the mathematical study of language possible.[8] These developments made considering infinite sets of sentences natural.

Not only was its importance not realized, the assumption that all instances of the consequence relation are knowable follows from traditional assumptions which are no longer as natural as they once were. The most common view has been that the truths of logic and mathematics are necessary. On this traditional view, if a sentence is necessary then its truth depends only on the concepts employed, that is, it is analytic. Further, if a truth is analytic then by reflection we can come to know it. The reasoning was that since an analytic truth depends only on the relations between the concepts employed, if we understand it then we lack nothing that is necessary in order to discover its truth. All that is required is the ability to think rationally about the concepts we possess. Hence, the traditional view held that all necessary truths can be known.

That this assumption might be too optimistic became evident with Gödel's Incompleteness results for arithmetic, thought to be the paradigm of a discipline consisting of necessary truths. Gödel showed that the first order arithmetic truths do not form a recursively renumerable set. If only a recursively enumerable set of sentences is knowable, then the Incompleteness Theorem shows that there are necessary truths which cannot be known.[9]

Of course, one can escape this conclusion by denying that knowledge is limited to recursively enumerable sets of sentences, but this only weakens the case for completeness. If we maintain that all necessary truths can be known, then second order logical truths, being necessary truths about sets, can also be known. We cannot criticize L2 for having unknowable logical truths. If, on the other hand, we argue that there are unknowable second order logical truths because only a recursively enumerable set of sentences can be known, then the Incompleteness Theorem shows that we cannot know all necessary truths. We then have no grounds for claiming that all logical truths can be known. In the first case, we cannot show that unkonwable second order logical truths exist; in the second, we cannot show they are a liability.

A defense of completeness considerations in choosing a logic need not hold that completeness is a necessary condition on any acceptable logic nor must it employ the categories of necessary and a priori truth. Another approach would be to treat completeness as a methodological constraint on logical theories, functioning as simplicity is usually claimed to function in accounts of scientific methodology. We do not know a priori that theories exceeding some specified degree of complexity acording to some accepted measure of simplicity are false. Before formulating and testing alternative theories, we cannot tell how complex the correct theory will be. Yet simplicity functions as a substantive constraint on theory choice. At the very least, if two theories fare equally well on all evidence and on other methodological criteria, then we should adopt the simpler of the two. Many think simplicity is so important that we should sometimes choose the simpler of two theories even if it isn't as well supported on other grounds.

We can view completeness in this way. Although we cannot demonstrate a priori that logic is complete, completeness could be held to have a certain weight in decisions between logics, so that we should only adopt an incomplete logic if it has advantages sufficient to outweigh its incompleteness. Various positions are possible on the precise status of completeness, depending on the importance accorded to it in relation to the other factors relevant in choosing a logical theory.

Quine seems to view completeness in this way. This approach is consistent with his rejection of a priori truth and is suggested by his treatment of Henkin's logic allowing branching quantifiers.[10] Quine cites reasons why one might choose this logic over first order logic. These reasons amount to the argument that for certain sentences of natural language, there are formulas of Henkin's logic that capture their intuitive meaning better than any formulas of first order logic. He then claims that there is an important consideration against adopting this logic, viz., it is incomplete. In 'Existence and Quantification,' he says that its incompleteness does not prove that Henkin's logic

provides an incorrect account of logical form, but the completeness of first order logic is an important consideration in its favor.[11] In his second discussion Quine goes farther, arguing that--despite the greater ability of Henkin's logic to capture the intuitive meaning of sentences of natural language--its incompleteness is a sufficiently important liability to justify rejecting it in favor of first order quantification theory.[12] In the same way, we might ignore a few disconfirming observations and retain a simple theory over a complex rival having no disconfirming evidence.

The view that completeness is a methodological constraint on logical theories seems promising until we try to justify this constraint. Philosophers have provided various justifications for methodological canons, but these types of justification do not support a completeness constraint. One can try to justify a methodological canon by appeal to intuition, e.g., claiming that it is self-evident that simple theories are preferable to complex rivals, all things being equal. But this type of justification can only be convincing if the canon is generally accepted. Then one might argue that it is self-evident and requires no further justification. This approach is attractive in justifying simple rules of inference on which there is complete agreement. When a canon is controversial, appeals to intuition do not help since those rejecting the canon do not find it to be evident. This is the situation with completeness, where some defend its importance while others think it is irrelevant.

One can also justify methodological canons either inductively or as a codification of actual practice. An inductive justification employs currently accepted scientific theory to show that the methodological canon generally leads to the truth. We might argue that past applications of this canon led to the adoption of theories that later received further confirmation. In discussing simplicity, Quine has argued that frequently there are fundamental biases favoring simpler hypotheses, both in the experimental criteria of concepts and in the deviations

tolerated in the experimental results taken to confirm an hypothesis.[13] In this way, one can attempt a 'scientific' justification of a canon of scientific methodology.[14]

In justifying a methodological canon as a codification of actual practice, one argues that the canon is one of the rules we actually follow when adopting theories. Quine also advocates this type of justification of scientific methodology. He has often emphasized that scientific method is but a refinement of common sense, with which it is continuous. When discussing simplicity, he invariably points out that considerations of simplicity are employed constantly, both in scientific and in common sense reasoning. Quine takes this approach to its logical conclusion in 'Epistemology Naturalized.'[15] He argues that epistemology is simply a part of psychology, being the study of how a human subject, starting from an initial input, arrives at a theory of the world. The aim is still the traditional epistemological one of determining how evidence relates to theory.

Though these types of justification are often attractive, there seems to be no hope of defending completeness in either way. The problem is that we do not have a sufficiently rich history of choices between formal logics. Unlike simplicity, completeness is not a consideration in common sense reasoning. As for scientific reasoning, only in the twentieth century could one formulate the question of the completeness of a logical theory. Logical systems were not presented formally before Frege and even he did not provide a semantic definition of logical consequence for his systems. Many formal logics have been studied in this century, but only the theory of types and first order logic ever were generally accepted. This hardly shows a continuing practice supporting completeness as a methodological canon. The theory of types, in the 1930's commonly thought to be the correct logic, is not complete. First order logic is complete, but its general adoption provides, at best, a single example of the use of completeness considerations. And in lieu of further choices of complete over incomplete logical systems,

we have no reason to think that completeness was, or should have been, a factor in the widespread adoption of first order logic. For the same reason, we have little evidence for an inductive justification. The choice of first order logic over its incomplete competitors is insufficient evidence to support an inductive justification of completeness.[16] In any case, citing the truth of first order logic as evidence for a completeness constraint requires rejecting incomplete alternative logics. To reject these logics because of their incompleteness is clearly circular.

There remains a more convincing defense of the importance of completeness in choosing between logical theories. This defense concerns the inference problem. If a logic has a recognizably complete proof procedure, then we know that this proof procedure solves the inference problem. The adoption of at least certain incomplete logics, in particular L2, puts us in an unusual situation. A consequence of these logics is that we can never recognize a solution to the inference problem. We argue that this is a reason, all things being equal, to prefer recognizably complete logics at least to incomplete logics having such undesirable consequences. We then finish this chapter by indicating our reservations about this conclusion.

The inference problem is to codify all inferences, that is, all knowable instances of the consequence relation. Employing various idealizations and assumptions, we identified this problem with that of finding a recursive proof theory generating all inferences. If we know that a proof procedure P for a logic L is complete, then P can be seen to solve the inference problem for L. Let Prov be the relation of provability by means of P and Inf be the inference relation, that is, those instances of the consequence relation that can be known. Knowing that P is complete, we know that Prov is a subset of Inf, and the inference relation is clearly a subset of the consequence relation. Since we know that any instance of the consequence relation is provable, Prov and Inf are known to be coextensive. An example is first order logic where

we have recognizably complete proof procedures which therefore solve the inference problem.

An example of an incomplete logic with undesirable consequences for the inference problem is L2. If a logical theory based on L2 is correct, then we can never recognize a solution to the inference problem. Actually, we show a stronger claim to be true. If P solves the inference problem, that is, generates all knowable instances of the consequence relation, then we cannot even recognize that P is consistent.[17] Suppose P is a proof procedure for L2, PA2 is the conjunction of the second order Peano axioms for arithmetic, GN(X) is the Gödel number of X and Con(n) is the arithmetical assertion that the proof procedure with Gödel number n is consistent. We assume that P is presented in the standard manner so that we can calculate its Gödel number. Let T1 be first order Peano arithmetic and T2 be the system of second order arithmetic which results by adding PA2 as a nonlogical axiom to proof procedure P. Finally, assume that we know P proves precisely those formulas known to be second order valid. Since P only proves valid formulas, we know

(1) P is sound.

PA2 is known to only have models that are isomorphic to the standard model of arithmetic. Since all arithmetic truths hold in this model, we know

(2) $PA2 \rightarrow A$ is second order valid if A is a truth of arithmetic.

Our knowledge of the truths of first order arithmetic and our assumption that P codifies our knowledge of second order logic shows that

(3) If $\vdash_{T1} A$ then $\vdash_{P} PA2 \rightarrow A$.

Since PA2 is an axiom of T2, (3) implies

(3') If $\vdash_{T1} A$ then $\vdash_{T2} A$.

From (1) and the truth of PA2 we have

(4) T2 is consistent.

Then (4) and our knowledge of Gödelization yields that

(5) Con(GN(T2)).

Thus, from (2) and (5) we know that

(6) $PA2 \rightarrow Con(GN(T2))$

is a valid formula of second order logic. Yet (6) is not provable by P. If it were, then in T2 we could prove Con(GN(T2)). But since (3') shows that T2 contains elementary arithmetic, we know by Gödel's Theorem that this is impossible. Since we know that any solution to the inference problem is consistent, we cannot recognize a solution if a second order account of logical consequence is correct. The adoption of L2 condemns us to ignorance concerning one of the fundamental problems facing a logical theory.

The assumption that only a recursively enumerable set of sentences can be known played a crucial role in these conclusions. We employed this assumption in our construal of the inference problem as the task of finding a recursive proof procedure generating all inferences. If we could recognize a nonrecursively enumerable set of sentences to be valid, and hence true, then a nonrecursive proof procedure would be necessary to codify the inference relation. We have not shown that such a nonrecursive proof procedure could not be reognized to solve the inference problem. In addition to excluding nonrecursive proof procedures, this assumption implies that the inference problem does have a solution. Since each recursively enumerable set is generated by a recursive proof procedure, there is a proof procedure P generating all inferences. If a second order account of logical form is correct, however, we can never recognize that P does this.

We have not shown that first and second order logic are typical. There may be complete logics where we cannot recognize a complete proof procedure or even know that one exists. All we showed was that if a proof procedure is <u>known</u> to be complete, then we also know that it solves the inference problem. Similarly, we didn't claim that all incomplete logics put us in the undesirable situation L2 does. There may be incomplete logics having proof procedures we can see codify all knowable instances of their consequence relations. Their incompleteness would be no liability. Thus, we have not drawn the general conclusion that completeness is a virtue and incompleteness a vice.

The existence of a first order proof theory known to be complete and the curious consequences of L2 distinguish this case. We feel that the undesirable consequences of L2 are a liability--we require a convincing argument before we will believe we are in such a self-defeating epistemological situation. Apart from general arguments for a logical theory based on L2, however, we have no reason to believe our situation is as self-defeating as second order logical theories would have it. These consequences make a second order account of logical form less plausible. Proof procedures known to be complete show that first order logical theories have more intuitive epistemological consequences.

These intuitions suggest a methodological canon counseling us to prefer the more epistemologically desirable of a pair of theories. That is, if theory A has the consequence that we can know more than we could if a rival theory B were true, this is a consideration in favor of A. It is not conclusive--B may have assets or A have liabilities sufficient to offset this advantage. This canon justifies our taking a recognizably complete proof procedure for first order logic as an advantage when we compare it to the undesirable consequences of L2.

The epistemological desirability of completeness is sometimes cited in defense of favoring complete to incomplete logics. Usually, however, the argument suggested is not the one we have presented. Our argument did not show that complete logics are always epistemologically preferable to incomplete rivals. Our more modest conclusion was that the existence of a proof procedure known to be complete favored a logic to one having the consequence that we can never recognize a solution to the inference problem. First and second order logic provide examples of such systems. Further argument is needed to establish that they are typical of complete and incomplete logics generally.

One might misconstrue the epistemolgoical consequences of first and second order logic by emphasizing our inability to recognize all instances of the second order consequence relation. Restricting ourselves to the case of logical truths, all first order logical truths can be

known via a recognizably complete proof procedure. Assuming that only a recursively enumerable set of sentences can be known, if L2 is correct then there are logical truths unknowable in principle. First order logic might seem preferable since it has the conseqeuence that all logical truths are knowable.

The reasoning is faulty since the class of logical truths depends on the correct logical theory. The unknowable logical truths of a second order logical theory remain unknowable, though nonlogical, truths if a first order account of logical form is correct. As we saw in Chapter II, given a proof procedure for one logic employing only axioms known to be true, there is a proof procedure of the same type for the other proving formalizations of precisely the same sentences. Any sentence known by means of a proof in one logic can be known by a similar proof in the other. First and second order logic differ on which of these sentences are logically true but not on which can be known. The epistemological consequences of these logics do favor first order logic, but not because there are unknowable logical truths if a second order account of logical form is correct. First order logic has an advantage because a first order account of logical form implies that we have a solution to the inference problem, while on a second order account we can never recognize a solution.

These conclusions are tentative. To reject second order accounts of logical form because of their undesirable epistemological consequences would require more extensive discussion than we have provided. We have already mentioned the importance of the assumption that only a recursively enumerable set of sentences can be known. This assumption is plausible, but a reasoned defense of it is necessary.

We have argued that the undesirable epistemological consequences of L2 are a liability. To turn this claim into an argument against logical theories based on L2 we must be more precise concerning the importance of epistemological consequences in theory choice. Presumably they are not conclusive. We have no *a priori* assurance that we are not in an

undesirable epistemological situation. Epistemological consequences are, at best, one factor to be considered when adopting a theory. We will only reject a theory with undesirable epistemological consequences in favor of a more desirable alternative if this consideration is not outweighed by others. To reject L2 for this reason we must state precisely what weight should be accorded to epistemological consequences and then show that on balance these consequences tip the scales against it.

A better argument is also needed for the claim that the epistemological consequences of first and second order logic are even relevant when choosing between them. We defended this claim by our intuitions concerning our epistemological situation and suggested a methodological canon implying it. Each needs a justification. Concerning this proposed methodological canon, we must show that the more epistemologically desirable of two theories is more likely to be true. This may be just wishful thinking. This canon seems to presuppose that one can argue from the desirability of a state of affairs to its reality. Is this any more justified when considering scientific theories than when discussing social systems? A defense must answer such objections.

If all this could be done, we would have an argument showing that second order accounts of logical form are incorrect. We will not pursue this line of argument since an easier path is open. In the next chapter we will present a different argument to show that a second order account of logical form is unacceptable. This argument is convincing and part of its persuasiveness comes from its uncontentious assumptions. It is neutral with respect to controversial issues of epistemology and scientific methodology such as those we have just mentioned. We will also see that, unlike a recent criticism of L2, it doesn't presuppose a solution to a complicated dispute on the nature of set theory. This argument will justify our rejecting second order accounts of logical form.

NOTES

1. This criticism is usually associated with Quine, yet he never criticizes L2 in this way. He only cites incompleteness when criticizing a logic allowing branching quantifiers, and then his criticism is more tentative than one might have expected. See 'Existence and Quantification' in Ontological Relativity and Other Essays (New York, 1969), pp. 108-113 and Philosophy of Logic (Englewood Cliffs, 1970), pp. 89-91.

2. Often one assumes that a logic comes with a proof procedure, and it is called complete if that particular proof procedure is complete. The definition we have given is then taken as a definition of a logic being completeable.

3. Sometimes a distinction is made between strong and weak completeness. The definition in the text is then a definition of strong completeness, while a logic is weakly complete if there exists a proof procedure such that a formula is provable if and only if it is valid.

4. See Chapter I, pp. 15ff.

5. We will often discuss the case of validity to simplify the exposition.

6. One could deny that only a recursively enumerable set of sentences is knowable and still maintain that there are unknowable logical truths in L2 but not in first order logic. As before, the existence of a recognizably complete proof procedure for first order logic shows that all first order logical truths are knowable. We might deny that all second order logical truths are knowable because of its close relationship with set theory. The truth of various open question of set theory is known to be equivalent to the validity of certain second order formulas. For example, substitute for the Replacement Schema of ZF the following second order sentence:

$$(\forall R)[(\forall u)(\exists ! v)R(u,v) \rightarrow (\forall z)(\exists x)(\forall y)(y \varepsilon x \leftrightarrow (\exists w)(w \varepsilon z \wedge R(w,y))].$$

Let the resulting second order theory be called ZF^2. Then the continuum hypothesis (CH) of ZF is true if and only if $ZF^2 \rightarrow CH$ is a valid formula of L2. We might argue that the truth or falsity of CH is unknowable and thus that there are unknowable logical truths in L2. The issue remains whether we have reason to believe that all logical truths can be known.

7. 'A consequence is a hypothetical proposition composed of an antecedent and consequent by means of a conditional connective or one expressing a reason which signifies that if they, viz., the antecedent and consequent, are formed simultaneously, it is impossible that the antecedent be true and the consequent false.' Quoted in I.M. Bocheński, A History of Formal Logic, trans. Ivo Thomas (New York, 1970), p. 190.

8. Although the invention of formal languages was necessary for the mathematical study of language, there is no trace of a mathematical treatment of language in Frege's work. Frege created formal languages in order to increase the certainty of mathematical results, but he never treated these languages as mathematical objects. The consideration of infinite sets of sentences became prominent in the early work in model theory, in particular in Skolem's extension of Löwenheim's theorem. See Skolem's 'Logico-Combinatorial Investigations in the Satisfiability or Provability of Mathematical Propositions: A Simplified Proof of a Theorem by L. Löwenheim and Generalizations of the Theorem' in _From Frege to Gödel_, pp. 252-63.

9. One could still maintain that if a necessary truth can be known at all, then it can be known _a priori_. Saul Kripke has made a persuasive criticism of these traditional connections between necessity, analyticity and a prioricity in 'Naming and Necessity' in _Semantics of Natural Language_, ed. Donald Davidson and Gilbert Harman (Dordrecht, 1972), pp. 253-355. The more limited criticism in the text suffices for our purposes.

10. See note 1.

11. _Ontological Relativity_, pp. 112-3.

12. _Philosophy of Logic_, pp. 90-1.

13. 'On Simple Theories of a Complex World' in _The Ways of Paradox and Other Essays_ (New York, 1966), pp. 242-5.

14. Such justifications raise the problem of circularity, for we are justifying a canon by using accepted scientific theories which may in turn receive some of their support from the canon in question. Since we will argue that completeness cannot be defended inductively, this problem will not concern us.

15. In _Ontological Relativity_, pp. 82ff.

16. This does not show that an inductive justification of a more abstract type is not possible, but we do not know of one.

17. D.A. Martin brought this argument to my attention.

CHAPTER V

In this final chapter we show that logical theories based on L2 provide an incorrect account of logical form. If quantification over the subsets of some set X is construed as second order quantification over X, we cannot formalize sentences containing quantification over sets containing subsets of X. To do this, quantification over subsets of X must be first order. Since we have the ability to quantify over ever larger totalities, no formalization employing second order quantification can be correct. This criticism of L2 is then shown to be more compelling than others due to its uncontentious assumptions. We conclude by considering whether other higher order logics are subject to the same criticism.

To see that second order logical theories provide an incorrect account of logical form, suppose that

(1) 'There exists a set P of natural numbers containing 0'

is correctly formalized in L2 by

(2) $(\exists P)\ P\overline{0}$,

where ω is the domain and $\overline{0}$ denotes 0. Now consider various contexts in which (1) might occur. It might appear in talk about natural numbers and sets of natural numbers. Formalizing this discourse in a second order language poses no problem. We simply add relation, function and constant symbols with appropriate interpretations. Quantification over the natural numbers is represented as first order and quantification over sets of natural numbers as second order quantification over ω.

But (1) could also occur in a discourse involving quantification over sets containing sets of natural numbers. (1) might be followed by

(3) 'If any set P of natural numbers contains 0,
then there is a set Q containing P.'

A natural second order formalization of (3) is

(4) $(\forall x)[(Cx \wedge \overline{0}\varepsilon X) \rightarrow (\exists Q)Qx]$,

where $\omega \cup \mathcal{P}(\omega)$ is the domain, $C = \mathcal{P}(\omega)$, ε is the membership relation restricted to $\omega \times \mathcal{P}(\omega)$ and $\overline{0}$ denotes 0.[1] We now have a problem.

Quantification over sets of natural numbers is interpreted in (2) as second order quantification over ω, while (4) construes it as first order quantification over those objects in the domain satisfying Cx. To formalize both (1) and (3) we must change one of our formalizations. There is no reason why the assertion that a set contains 0 should have a different logical form in (1) than it has in the antecedent of (3).

That this is unacceptable is shown by our inability to represent instances of the consequence relation. Consider

(5) 'If any set P of natural numbers contains 0, then there is a set Q such that Q contains P and has at least two members,'

which we can formalize by

(6) $(\forall x)[(Cx \wedge \overline{0}\varepsilon x) \rightarrow (\exists Q)(Qx \wedge (\exists y)(y \neq x \wedge Qy))]$.

(1) and (5) clearly imply

(7) 'There exists a set with at least two members,'

but (2) and (6) don't imply

(8) $(\exists Q)(\exists x)(\exists y)(x \neq y \wedge Qx \wedge Qy)$.

In the interpretation $\langle \{a\}, F \rangle$, where $F(\overline{0}) = a$ and both $F(C)$ and $F(\varepsilon)$ are the empty set, (2) and (6) are true while (8) is false. We cannot show that inferring (7) from (1) and (5) is an instance of modus ponens unless (1) and the antecedent of (5) receive the same formalization. Quantification over $\mathcal{P}(\omega)$ must be uniformly treated as either first order or second order. Either (1) or else (3) and (5) have been incorrectly formalized.

Our only option, however, is to revise our formalization of (1) since any formalization of (3) must construe quantification over $\mathcal{P}(\omega)$ as first order. A formalization of (3) will treat quantification over sets which contain sets of natural numbers as either first order or second order. If treated as first order, the domain will be a superset of $\mathcal{P}(\mathcal{P}(\omega))$. (3) might be formalized by

(9) $(\forall x)[(Cx \wedge \overline{0}\varepsilon x) \rightarrow (\exists y)(x \varepsilon y)]$,

where the domain is $\omega \cup \mathcal{P}(\omega) \cup \mathcal{P}(\mathcal{P}(\omega))$, C and $\overline{0}$ are as before and ε is the

membership relation restricted to the domain. As this formalization illustrates, if quantification over $\mathcal{P}(\mathcal{P}(\omega))$ is first order then quantification over $P(\omega)$ must be first order as well.

If we drop (4) in favor of another second order formalization, the result is the same. If we construe quantification over sets containing sets of natural numbers as second order, then to formalize the assertion that a set Q contains some set x of natural numbers we must see quantification over $\mathcal{P}(\omega)$ as first order and give this assertion the form Qx. Whatever the correct formalization of (3), quantification in (1) must be interpreted as first order. (2) is therefore an incorrect formalization of (1).

This argument doesn't depend on any special properties of (1). An analogous argument shows that any formalization containing a second order quantifier is incorrect. Consider any sentence with the form

(10) . . . there exists a subset P of D such that B . . .

and suppose that we construe this as second order quantification over D. Now consider

(11) 'There exists a subset P of D and there exists a set Q such that B and Q contains P.'

To formalize (11) we must abandon our formalization of (10). As we showed with (1) and (3), whether we construe Q as a first order or as a second order variable, quantification over $\mathcal{P}(D)$ must be first order. Only in this way can we represent the logical form of both (10) and (11) in a second order language.

For any particular sentence, a second order account of logical form may hold that all quantification in this sentence is first order. But for a second order logical theory to be correct, there must exist sentences employing second order quantification. For some D quantification over $\mathcal{P}(D)$ must be correctly formalized as second order. Otherwise, all sentences receive a correct formalization in the first order sublanguage of L2 and thus first order logic is adequate to represent the logical form of all sentences of natural language. The addition of predicate variables is superfluous.

But whenever quantification over subsets of some D is taken to be second order we are unable to formalize quantification over sets containing these subsets. To provide a general account of the logical form of natural language, quantification over subsets of any set D must be formalized as first order. Since no sentence is correctly formalized by a sentence containing a second order quantifier, a second order account of logical form is incorrect.

One might reply to this criticism by arguing that a second order account of logical form is correct but that the infinitely many sentences correctly formalized using second order quantifiers are ambiguous. Perhaps in some contexts an utterance of (1) 'There is a set P of natural numbers containing 0' has the logical form of (2) $(\exists P)\ P\overline{0}$ and in others the logical form of the antecedent of (4), that is,

(12) $(\exists x)(Cx \wedge \overline{0}\varepsilon x)$.

This would differ from the usual examples of ambiguity since if (1) is ambiguous between (2) and (12), then in some contexts it is valid while in others it is not.

Yet this view cannot be maintained. If (1) is ambiguous, then there must be some objective feature of an utterance of (1) in virtue of which it has the logical form it does. The only remotely plausible candidate here is the speaker's intentions. But when uttering (1) speakers do not intend their quantification either to be first order or to be second order. They merely intend to assert that there is a set of natural numbers containing 0, and this intention is compatible with either reading of (1). (2) and (12) have the same truth conditions. Under the interpretations given them, both state that there is a set of natural numbers containing 0. Yet they differ in their logical form. (2) is valid but (12) is not. For any set D as domain and any y in D taken as the denotation of $\overline{0}$, there exists a member of $\mathcal{P}(D)$ containing y. (2) is therefore true under every interpretation and is valid. (12) is true under the interpretation provided but is false if, e.g., we take the extension of C to be the empty set. It is a nonlogical truth.

But whatever intentions a speaker may have in uttering (1), (2) is not a correct formalization of this utterance. If someone replies (3) 'If any set P of natural numbers contains 0, then there is a set Q containing P,' then to formalize this discourse in a second order language we cannot take (2) as our formalization of (1). This is so, regardless of the intentions of the speaker uttering (1).

Here one can't argue that the second speaker misunderstood the first by taking (1) in the sense of (12). Both attach the same intuitive meaning to the first speaker's utterance, viz., the claim that there is a set of natural numbers containing 0. An indication of this is that any speaker uttering (1) would see (3) as an appropriate response. Both would also acknowledge that (1) and (5) imply (7), though this instance of the consequence relation can only be represented if we abandon (2) as our formalization of (1).

We have argued that quantification over the powerset of any set is never correctly construed as second order. One could agree and yet try to defend L2 by maintaining that second order quantification only occurs when we are quantifying over all properties of the universe V of all sets. Only here, one might argue, do we truly treat the values of our predicate variables as properties and as not subject to some later first order quantification.

But why should we believe in properties of sets? The usual reason given is that quantification over properties of sets is necessary to provide a truth definition for ZF. We believe that the theorems of ZF are true, but we only have an informal account of what this means. We might try to avoid the concept of truth by always asserting, e.g., that there exists the powerset of any set rather than saying that the powerset axiom is true. But our belief in the Replacement Schema can only be expressed by saying that each of its infinitely many instances is _true_.[2] So even to express our belief in ZF, we need the concept of truth. Yet a truth definition for ZF requires quantification over properties of sets. The present view is that only quantification over such properties is truly second order.

Although the solution of the problem of defining truth for statements of set theory does require quantification over properties of sets, the notion of property employed in second order logic does not allow a satisfactory solution. The concept of property in second order logic is that of an arbitrary collection and precisely corresponds to the concepts of subset and subclass. If V is the domain for a second order language, then the predicate variables must range over $\mathcal{P}(V)$, the powerclass of V. Like Kelley-Morse set theory, second order ZF is intended to describe the structure $R_{O_{n+1}}$ in which V and $\mathcal{P}(V)$ are added as two final stages to the cumulative hierarchy.[3]

But we know that the quantification over classes of sets occurring in Kelley-Morse or in second order ZF does not really allow us to solve the problem of truth for set theory--it is merely shifted to these new theories. In both second order ZF and Kelley-Morse a truth predicate for (first order) ZF can be defined, but then we have the problem of defining truth for them. This requires a still more powerful theory quantifying over all properties of $R_{O_{n+1}}$.

If the need to quantify over properties of sets to provide a truth definition for ZF is a reason to believe in properties of sets, we have equal reason to believe in properties of $R_{O_{n+1}}$. But quantification over these properties, that is, over $\mathcal{P}(\mathcal{P}(V))$, shows that quantification over $\mathcal{P}(V)$ is not second order after all. Whether we treat quantification over $\mathcal{P}(\mathcal{P}(V))$ as first order or as second order, a sentence like 'There exists a P in $\mathcal{P}(V)$ and a Q in $\mathcal{P}(\mathcal{P}(V))$ such that P is a member of Q' can only be formalized in a second order language if quantification over $\mathcal{P}(V)$ is first order. Quantification over properties is necessary to solve the problem of truth for set theory, but the requisite notion of property is not that of arbitrary collection. To treat properties of sets as arbitrary collections or classes neither allows a satisfactory solution to this problem nor avoids the conclusion that second order formalizations are incorrect.

Our ability to quantify over ever larger totalities shows that no quantifier in a sentence of natural language is correctly regarded as second order. A first order account of logical form does not have these difficulties. Given a set X of sentences to be formalized in first order logic, we choose a domain D containing the denotations of all names used in X and containing all objects that are quantified over or are in the extension of any predicate occurring in a sentence in X. If we then want to represent the logical form of other sentences quantifying over objects not in D, e.g., sentences quantifying over $\mathcal{P}$(D), we simply enlarge the domain to include these objects and add whatever new nonlogical symbols are needed. Quantification over D remains first order. A first order account of the logical form of the sentences in X can thus be extended. This piecemeal approach to an account of logical form allows us to represent the logical form of ever larger portions of natural language.

In contrast, a second order account cannot be consistently extended. As in first order formalizations, when formalizing a set of sentences in second order logic we choose a domain D. Quantification over D is then interpreted as first order and quantification over subsets of D as second order. But this account cannot be extended to sentences quantifying over sets which contain subsets of D. To formalize such sentences we must see quantification over subsets of D as first order. Second order quantification cannot be part of a general account of the logical form of natural language.

Part of the persuasiveness of this criticism is that it doesn't rest on controversial assumptions. In the last chapter we considered a criticism of L2 based on its epistemological consequences. Though plausible, this criticism assumed controversial views concerning the limits of human knowledge and the nature of scientific methodology. This criticism is unconvincing to the extent that these views are dubious. The criticism we have just presented, however, does not presuppose controversial positions in epistemology or the philosophy of

science. Further, unlike a recent criticism of L2 by George Boolos,[4] it is also neutral with respect to a complex dispute over the nature of set theory.

Boolos argues that we cannot formalize assertions about all sets in a second order language. Knowing Russell's Paradox, we might find the validity of a second order set theoretic sentence like

(13) $(\exists P)(\forall x)[Px \leftrightarrow \sim(x \varepsilon x)]$

surprising. But the model theoretic definition of validity for L2 only requires that (13) be true when any set is chosen for domain. Given any set D as domain and interpreting ε as the membership relation, there does exist a subset of D containing all non-self-membered sets. Boolos argues that since Russell's Paradox shows that there exists no set of all non-self-membered sets, the universe V of all sets cannot be taken as a domain for L2. If we took V as a domain for (13) and took P to range over all sets of objects over which x ranges, then (13), though valid, would be false.[5]

Boolos concludes that L2 is limited by not being able to provide formalizations of assertions about all sets, such as occur in set theory. First order logic, on the other hand, is not restricted in this way. Although the model theory for first order logic requires an interpretation to be a set, Boolos argues that non-set theoretic interpretations are intelligible. He cites first order ZF whose intended interpretation is $\langle V, E \rangle$, where E is the membership relation. Although this interpretation is not a set, he says the axioms of ZF formalize true assertions about this structure. Since non-set theoretic interpretations of L2 are not permissible, second order formalizations cannot capture the meaning of such set theoretic statements.

This criticism is problematic since it presupposes a particular solution to a controversial dispute on the nature of set theory. The presupposition is that the domain of ZF is V and that this is a permissible domain for the first order quantifiers. 'ZF (Zermelo-Fraenkel

set theory) is couched in the notation of first order logic, and the quantifiers in the sentences expressing the theorems of the theory are presumed to range over all sets, even though (if ZF is right) there is no set to which all sets belong.'[6] This is the most common view of set theory. On this view, which sets exist and whether one set is a member of another is completely determinate. Accordingly, we can take the sentences of ZF at face value, viz., as assertions about $\langle V,E \rangle$.

Opposed to this view are a cluster of views receiving increasing attention.[7] Common to them is the view that first order quantification over V is not coherent since V is not a determinate totality. This rests on two claims: one about the first order quantifiers and the other about V. The first is that any domain D for the first order quantifiers must be a determinate totality, that is, which objects are in D must be objectively determined. We may not know what objects are in D--we can quantify over collections of which we know little--but the meaning of the first order quantifiers makes it clear that the domain must be determinate. A sentence of the form $(\forall x)A(x)$ is true under an interpretation just in case every object in the domain satisfies $A(x)$ when assigned to x. This condition only specifies a determinate truth value if the domain is a determinate totality.

This claim is not controversial, but the second one is. The second claim is that the totality of sets is not a determinate, but merely a potential, totality. V is commonly agreed to have the structure of a cumulative hierarchy whose stages are indexed by the ordinal numbers. The present view maintains that the length of this cumulative hierarchy is 'absolute infinity' since the ordinals exhaust our notions of infinity. That is, whenever there is a determinate sequence of stages of the hierarchy then the length of this sequence is given by an ordinal number. Since no ordinal number measures the length of the hierarchy itself, V does not have a determinate length and thus is not a determinate totality.

If this view is correct, then V cannot serve as a domain for the first order quantifiers and a new account of the meaning of set theoretic statements is needed. Charles Parsons argues that we should continue to accept ZF, claiming that we can always interpret a person to be quantifying over a suitably large R_α when ostensibly making assertions about all sets.[8] Or one could claim that assertions about all sets are permissible but that they must be formalized in an intuitionistic logic.[9] Other views are possible. The important point for our argument is that on these views V cannot serve as a domain for the first order quantifiers.

The relevance of this dispute in assessing Boolos's criticism is clear. Boolos argues that only in first order logic can we formalize assertions about all sets. This criticism of L2 assumes that V is a determinate totality and thus an acceptable domain for the first order quantifiers. But if V is merely a potential totality, then it can no more serve as a domain for first order logic than it can for L2. All coherent interpretations of either logic would be sets. To evaluate Boolos's criticism we must resolve a fundamental controversy over the nature of set theory.[10]

Our criticism avoids this thorny problem--it is neutral with respect to the nature of V. In replying to the view that only quantification over all properties of V is truly second order we did assume that V was a determinate totality. But simply to take this view seriously we had to suppose that V was determinate and so could be taken as a domain for L2. By not presupposing such controversial views, our criticism of L2 is more compelling than either Boolos's or that considered in Chapter IV.

We have shown that second order logic cannot provide the basis for a correct account of the logical form of natural language, but what about a logic of some higher order? If we are ready to use second order quantification, we should have no objection to using third order or even higher order quantification if necessary. If quantification over $\mathcal{P}(D)$ is construed as second order, we could treat quantification over sets

containing sets in $\mathcal{P}(D)$ as third order. (3) might be formalized by

$$(\forall P)[P\bar{0}\to(\exists Q)Q(P)],$$

where the domain is ω and Q is a third order predicate variable.

To give a general account of higher order logics we need the notion of the cumulative hierarchy generated from a set of X, defined by

$$R_1(X) = X,$$

$$R_{\beta+1}(X) = R_\beta(X)\cup\mathcal{P}(R_\beta(X))$$

and

$$R_\lambda(X) = \bigcup_{\beta\,\lambda} R_\beta(X)$$

where β is any ordinal and λ any limit ordinal.[11] In any interpretation with domain D the (one-place) α^{th}-order variables of a higher order logic range over $R_\alpha(D)$.[12] But an analogous argument to that we gave against L2 shows that no higher order logic having predicate variables of all orders less than some ordinal α (greater than 2) can be correct. Suppose that quantification over $R_\beta(D)$ for β less than α has been construed as β^{th}-order. Then how do we formalize sentences containing quantification over $R_\alpha(D)$? It would be natural to construe such quantification as α^{th}-order, but we are supposing that there are no α^{th}-order variables, only variables of order less than α. In order to formalize such sentences we must change our formalization of sentences quantifying over $R_\beta(D)$ for some β less than α so that we have not exhausted all orders of variables when we come to formalize sentences quantifying over $R_\alpha(D)$. If $\alpha=\beta+1$ for some β, then quantification over $R_\beta(D)$ cannot be of order β as claimed. Quantification over $R_{\beta+1}(D)$ must be β^{th}-order or lower and so quantification over $R_\beta(D)$ is lower still. If α is a limit ordinal, then variables ranging over $R_\alpha(D)$ are of order β for some β less than α. Then for this β variables ranging over $R_\beta(D)$ must be of an order less than β, rather than of β^{th}-order as we originally supposed.[13] In either case, no sentence is correctly formalized as containing β^{th}-order quantification. The β^{th}-order variables--and those of higher order if α is a limit ordinal--are superfluous.

If we want to consider a logic of higher than second order, we must consider what we will call Onth-order logic--a logic containing variables of order α for each positive ordinal α. We do not have a conclusive argument against Onth-order logic, but we will conclude by sketching some of the issues involved. Onth-order logic has the logical constants of first order logic, denumerably many individual variables of order 1 and denumerably many one-place predicate variables of order α for every ordinal α greater than 1. As with L2, the <u>individual terms</u> are identical with the terms of first order logic. A <u>term</u> is either an individual term or a variable of order greater than 1. If R is an n-place predicate then $R(t_1, \ldots, t_n)$, $u_1=u_2$ and $u_1(u_2)$ are formulas for any individual terms $t_1, \ldots, t_n$ and any terms u_1 and u_2. A variable of any order may be bound by a quantifier.

In any interpretation of Onth-order logic with domain D, the α^{th}-order variables range over $R_\alpha(D)$. The interpretations of Onth-order logic are identical with the interpretations of first order logic. If D is a set of individuals,[14] the definition of truth under an interpretation $\langle D,F\rangle$ is analogous to that for L2, with the α^{th}-order variables ranging over $R_\alpha(D)$. If v is an α^{th}-order variable, in any interpretation a sentence of the form $(\exists v)\,A(v)$ is true if there is a set in $R_\alpha(D)$ satisfying A(v).

There are some differences between Onth-order logic and L2. The orders in Onth-order logic are cumulative while those in L2 are not. For example, in any interpretation $\langle D,F\rangle$ the second order variables of Onth-order logic range over $D \cup \mathcal{P}(D)$; the predicate variables of L2 range over $\mathcal{P}(D)$. Also, Onth-order logic doesn't impose any conditions of stratification on its formulas—t=u and u(t) are formulas for any terms u and t. In L2 t=u and v(t) are formulas only if t and u are individual terms and v is a predicate variable.[15]

These differences between Onth-order logic and our system L2 of second order logic suggest a criticism of our argument against second order logic. Perhaps our argument relied on the particular formulation we chose. A system allowing unstratified formulas might seem to avoid

our objection. In such a system (1) could be formalized by

(14) $(\exists P)[(\forall x)(Px \rightarrow Nx) \wedge P\overline{0}]$

and (3) could be formalized by

(15) $(\forall P)[(\forall x)((Px \rightarrow Nx) \wedge P\overline{0}) \rightarrow (\exists Q)Q(P)]$,

where the interpretation is $\langle \omega \cup \mathcal{P}(\omega), F \rangle$ and F is such that $F(N)=\omega$ and $F(\overline{0})=0$. Here Q and P are both second order variables. If (14) and (15) are acceptable formalizations of (1) and (3), then we can formalize (3) while still construing quantification over $\mathcal{P}(\omega)$ as second order. Thus, a second order logic allowing unstratified formulas might seem to escape the criticism we made of L2.

But the mistake here is that admitting unstratified formulas requires a change in semantics--the semantics for L2 are not appropriate for such a system. In providing a definition of truth under an interpretation for On^{th}-order logic we said that if the domain is a <u>set of individuals</u> then the definition is analogous to that for L2. This restriction is necessary, for otherwise there will be isomorphic interpretations satisfying different sentences. For example, let a and b be two individuals, P be a second order variable and $\langle \{a,b\}, F \rangle$ and $\langle \{a, \{a\}\}, F' \rangle$ be two isomorphic interpretations. Under a definition analogous to that for L2, $(\exists P)(\exists x)(P=x)$ would be true under the first interpretation but false under the second. The problem is that--if we allow unstratified formulas and the domain should contain sets--this definition doesn't treat these sets <u>as if</u> they were individuals. If D has the same cardinality as some set of individuals, then we can define truth under an interpretation $\langle D,F \rangle$ by saying that a sentence A is true in $\langle D,F \rangle$ if there is an isomorphic interpretation $\langle I,F' \rangle$, where I is a set of individuals, and A is true in $\langle I,F' \rangle$ under a definition analogous to that for L2.[16]

The relevance of this difference in semantics is that (15) under $\langle \omega \cup \mathcal{P}(\omega), F \rangle$ does not formalize (3) since (3) is true, while (15) is false under this interpretation. (15) is false since it would not be

true if the domain contained only individuals.[17] Quite generally, if we take Q and P to be second order variables, then

(16) Q contains P

cannot be formalized even in an unstratified second order logic. The reason is that in any interpretation whose domain is a set of individuals

(17) Q(P)

is not satisfied by any evaluation. Yet (16) is true for certain sets Q and P. Consequently, though (3) is true, (15) $(\forall P)[(\forall x)((Px \rightarrow Nx) \wedge P\overline{0}) \rightarrow (\exists Q)Q(P)]$ is false under every interpretation. Due to this difference in semantics between stratified and unstratified systems, our criticism of L2 is also a criticism of unstratified second order logic.

Can On^{th}-order logic provide correct formalizations of natural language? One relevant issue is whether the universe V of sets is a determinate totality or whether in proposing an On^{th}-order logic one is committed to this view. If so, then On^{th}-order logic is incorrect. Suppose that for each (positive) ordinal α quantification over $R_\alpha(D)$ is claimed to be α^{th}-order. If V is determinate then so is On, and this implies that $\bigcup_{\alpha\varepsilon On} R_\alpha(D)$ is a determinate totality. Quantification over $\bigcup_{\alpha\varepsilon On} R_\alpha(D)$ is therefore coherent. Yet sentences containing quantification over this totality cannot be formalized unless we change our account of the logical form of other sentences. Variables ranging over $\bigcup_{\alpha\varepsilon On} R_\alpha(D)$ will be β^{th}-order for some β and thus quantification over $R_\alpha(D)$ for any α greater than or equal to β can no longer be construed as α^{th}-order.

On^{th}-order logic would seem to presuppose that On is a determinate totality. True, any quantification expressible in this logic is quantification only over some particular $R_\alpha(D)$. But to specify the syntax and semantics of On^{th}-order logic we must quantify over On. When specifying its vocabulary we said that for all ordinals α there are denumerably many variables of order α. If On is not a determinate totality, then this quantification is illegitimate.

Perhaps one could avoid this conclusion by treating On^{th}-order logic itself as merely potential. Whenever a sentence of natural language quantifies over some particular $R_\alpha(D)$, this can be formalized as α^{th}-order quantification. Rather than contain variables of all orders, On^{th}-order logic might be held to be indefinitely extendable--for any particular ordinal α one can specify, it would have variables of that order.

Another problem is whether On^{th}-order logic is learnable. If it has On^{th}-many primitive symbols then clearly it is not. Since a logical theory based on this logic holds that natural language has the structure of an On^{th}-order language, this would be a conclusive argument against it. If On^{th}-order logic could be intelligibly construed as a potential totality providing the logical form of any particular sentence we might utter, then this criticism might be avoided. Rather than have On^{th}-many orders of variables, an On^{th}-order language would merely have a potentially infinite number of orders. Such a language might well be learnable.

As we add more orders of variables, higher order logics come to look more like set theory. Any true statement of ZF in which all quantifiers are restricted to particular stages of the cumulative hierarchy generated from the set I of individuals can be formalized in On^{th}-order logic by a logical truth. We simply construe quantification over each $R_\alpha(I)$ as α^{th}-order quantification. Having come this far in construing logic to resemble set theory, we should mention the final step in this direction which we will call L_V. Add to first order logic denumerably many one-place predicate variables and a two-place predicate symbol ε. The individual terms are those of first order logic. The predicate symbols other than ε and = apply only to individual terms, while $t\varepsilon u$ and $t=u$ are formulas for any individual terms or predicate variables t and u. Both individual and predicate variables may be bound by quantifiers. An interpretation of L_V is identical with an interpretation of its first order sublanguage, where ε is treated as a logical

constant and not assigned a meaning. If D is a set of individuals, then a sentence A of L_V is true under an interpretation $\langle D,F\rangle$ if it is true when the individual variables range over D, the predicate variables range over $\bigcup_{\alpha\varepsilon O_n} R_\alpha(D)$, ε is construed as the membership relation and the remaining predicates, function symbols and constants receive the meaning given them by F.

L_V is simply ZF with individuals expressed in the form of a logic. The intended interpretation of ZF with individuals is taken as the range of the predicate variables in every interpretation of L_V. Any true statement of ZF with individuals can be translated into a valid sentence of L_V. L_V has only denumerably many symbols and is clearly learnable. Although a logical theory based on L_V presupposes that the cumulative hierarchy generated from a set of individuals is a determinate totality, L_V has no difficulty in formalizing sentences quantifying over such hierarchies. L_V is a particularly clear formulation of the view that set theory is logic. If L_V is logic then the usual reductions of mathematics to set theory show that--as the logicists claimed--mathematics can be reduced to logic.

On^{th}-order logic and L_V have brought us a long way from second order logic, and they might seem far less plausible. In particular, many feel that logic should not have the enormous ontological commitments of set theory. What we have shown, however, is that any consideration of higher order logic must concern these systems. A logical theory based on second order logic--or on any higher order logic of less than On^{th}-order--cannot provide a correct account of the logical form of natural language. This conclusion does not constitute a general defense of first order logic, but it does show it to be superior to its oldest rival.

NOTES

1. We have glossed over a minor error here to simplify the example. (4) actually does not formalize (3) since in (3) Q ranges over all sets, while in (4) it ranges only over subsets of $\omega \cup \mathcal{P}(\omega)$. This problem would be avoided without affecting our argument if (3) and (4) were replaced by

> (3') 'If any set P of natural numbers contains 0, then there is a set Q of <u>sets of natural numbers</u> containing P'

and

$$(4') \quad (\forall x)[(Cx \wedge \overline{0}\varepsilon x) \rightarrow (\exists Q)(\forall y)((Qy \rightarrow Cy) \wedge Qx)].$$

2. D.A. Martin discusses this point in 'Sets versus Classes' (manuscript), p. 4.

3. On is the class of ordinals. The cumulative hierarchy is $\bigcup_{\alpha \varepsilon On} R_\alpha$, where R_0 = the empty set, $R_{\alpha+1} = \mathcal{P}(R_\alpha)$ and $R_\lambda = \bigcup_{\alpha<\lambda} R_\alpha$ for each ordinal α and limit ordinal λ. In considering R_{On+1} we imagine the ordinals extended.

4. 'On Second-Order Logic,' pp. 514-5.

5. Here we are assuming that there do not exist classes distinct from sets, i.e., that if x is a member of y then y is a set. If such proper classes do exist, then we could take V as a domain for L2 and let the predicate variables range over the powerclass of V. But Boolos points out that this only shifts the problem: "One of the lessons of Russell's Paradox is that if we read 'Xx' as '(OBJECT)X bears R to (object) x,' then the range of first order quantifiers in second--but not first--order sentences may not contain all OBJECTS" (p. 515).

Assuming that there exist no proper classes, we think the correct conclusion is not that certain valid sentences of L2 turn false when V is the domain, but rather than we can't even make sense of taking V as a domain for L2. We intend the predicate variables of L2 to range over all arbitrary collections of elements of the domain. Yet if there exist no proper classes, when V is the domain there exists no appropriate totality for the predicate variables to range over. Whichever is the correct conclusion, if there do not exist proper classes then V is not a permissible domain for L2.

6. Boolos, p. 515.

7. See Charles Parsons, 'Sets and Classes,' Nous 8 (1974), pp. 10-11; D.A. Martin 'Sets versus Classes' which criticizes Parson's position and Michael Dummett, Frege: Philosophy of Language (New York, 1973), Chapter 15.

8. Parsons, loc. cit.

9. Saul Kripke proposed this view in conversation.

10. Kreisel's criticism of formal semantics also assumes that V is a determinate totality. (See Chapter I, pp. 11-12). Kreisel argues that the usual model theoretic definitions of logical consequence diverge from our intuitive concept since they do not consider non-set theoretic interpretations, like $\langle V,E \rangle$. If any determinate totality is a set, however, then all interpretations are set theoretic. If this view is correct, then Kreisel has not shown that model theoretic definitions of logical consequence fail to explicate our intuitive concept.

11. Here the first stage of the hierarchy is $R_1(X)$ rather than $R_0(X)$ as is standard. We framed the definition in this way so that the first order variables would range over $R_1(X)$, the second order variables over $R_2(X)$ and so on.

12. This account of higher order logics diverges from our account of second order logic. The differences will be discussed shortly.

13. Similarly, for any γ such that $\beta \leq \gamma < \alpha$, quantification cannot be construed as γ^{th}-order. It must be δ^{th}-order for some δ less than β.

14. In a moment we will discuss the reason for this restriction.

15. Cumulative orders and unstratified formulas are necessary to avoid problems at limit ordinals.

16. Richard Montague discusses this problem in defining truth for higher order formulas in 'Set Theory and Higher-Order Logic' in Formal Systems and Recursive Functions, ed. J.N. Crossley and Michael Dummett (Amsterdam, 1965), pp. 145ff.

Montague rejects the definition we have just given as a general definition of truth since it would assume that given any set there is a set of individuals having the same cardinality. He provides a general definition by using the notion of a model of a system he calls 'rank-free set theory with individuals.' The definition in the text will suffice for our purposes.

Our account of On^{th}-order logic has followed Montague's.

17. We are treating the natural numbers as individuals.

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