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# Superstrlngs in Four Dimensions

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by

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Introduction

All of the work presented in this thesis[1-3] evolved from the initial idea that strings, if they are to make sense, ought to exist in four spacetime dimensions. At the time that it was formulated, back in mid-1985, such an idea was thought to be impossible. There were only five string theories that were known to be fully consistent,[4-9] and each of these had a critical dimension of 10. The bosonic (or Veneziano) string has to live in 26 spacetime dimensions, or else a breakdown of Lorentz invariance occurs. But it is not a fully consistent theory because it contains a tachyon (a particle with negative mass squared) in its spectrum[38-40] and its amplitudes have infinities. The open (type I) as well as the closed (type II) superstrings introduce supersymmetry into the theory, and this reduces the critical dimension from 26 down to 10. The type I superstring is N=1 supersymmetric, whereas the type II string is N=2. The $E_8 \times E_8$ and $Spin(32)/Z_2$ heterotic strings[10,4] are hybrids (or heteroses) of the 10-dimensional superstring and compactified versions of the 26-dimensional bosonic string, and they too have ten as their critical number of dimensions. All of these theories have very appealing features in their critical number of dimensions, and the hope was to maintain as many of these features as possible while reducing the spacetime dimension down to what we know it must be on the everyday scale of events - namely four.

The most attractive feature that all of these theories share is that they are only consistent if they include spin-2 particles, or gravitons, in their

interactions. Strings can occur in either of two configurations, open or closed. Open strings are free at the ends, while closed strings are joined at the ends and form loops. The interactions of open strings consists of their splitting and joining and necessarily involves the formation of closed strings when the two ends of an open string join together. Thus open string theories always include closed strings; however, closed strings need not contain open strings. It is in the massless sector of closed strings that gravitons arise, and it follows that since all string theories include closed strings, they all include gravity as well. This is what makes strings so compelling.

The Veneziano model was the first string theory to be worked out. It is a theory of bosons only that exhibits the features of Regge trajectories, crossing symmetry, and duality - things which themselves first led to the investigation of dual resonance models and string theories. While the Veneziano model (VM) is a modular invariant theory, it holds little interest as a candidate for unifying all the fundamental forces between particles because it contains a tachyon in its spectrum. Supersymmetric string theories, on the other hand, do not contain tachyons - their lowest lying states are massless and contain fermions in addition to bosons. Supersymmetry also has the added advantage that it eliminates the "hierarchy problem" in gauge theories which contain scalar particles (a Higgs sector). These theories are said to be "unnatural" in themselves since they require the fine tuning of parameters with an incredible accuracy. The only way to make such theories natural again is by introducing a new symmetry to constrain the parameters of the theory, and that is what supersymmetry does. By requiring that there be equal numbers of bosons and fermions at each mass level, the radiative corrections contributing to the hierarchy problem exactly cancel - thus eliminating the problem. In the string theories as well, before any

field theory limits have been taken, supersymmetry causes the one-loop cosmological constant to vanish identically, thus - assuming this holds to higher orders as well - supersymmetry eliminates the divergences which plague nonsupersymmetric string theories. Supersymmetry seems to be an essential part of string theories, and all of the phenomenologically interesting models are supersymmetric.

The most promising string theories appear to be the D=10 type I and type II superstrings and the heterotic string. The type I superstring and heterotic string were preferred back in 1985 because they could contain, in addition to gravity, supersymmetric gauge theories large enough to contain the standard model $SU(3)\times SU(2)\times U(1)$. This was acheived in one of two ways: for the open type I superstring the Chan-Paton[11] method of attaching charges to the ends of the strings was used, while the heterotic string employed the Frenkel-Kac-Segal (FKS) construction.[12-14] The Chan-Paton method can be used to introduce classical groups into the theory by associating matrices in the fundamental representations of these groups with the external string states of a given process. The closed string states which arise in type I theories are singlets under the Chan-Paton charges. The FKS method, on the other hand, distributes the charges along the length of a closed string by compactifying some of the bosonic string coordinates in a way that introduces a Kac-Moody algebra into the theory. The compactified coordinates are curled up to form the maximal torus of the corresponding Lie algebra, and the rank of the gauge group equals the number of compactified internal coordinates, which is 16 for the heterotic string. The group generators in this construction are represented by the vertex operators of the interacting theory. For the heterotic string, the consistency of the interactions does not allow for just

any rank 16 Lie group - only the choices $E_8 \times E_8$ and spin(32)/$Z_2$ give modular invariant one-loop amplitudes.

The heterotic string, with massless gauge bosons having either $E_8 \times E_8$ or SO(32) symmetry, emerged on the scene in late 1984, only months after Green and Schwarz had proved that in D=10 supersymmetric Yang-Mills theory coupled to N=1 supergravity there were anomalies that only cancelled if the gauge groups were either $E_8 \times E_8$ or SO(32).[15] Precisely those gauge symmetries singled out by the modular invariance of the interacting theory were the ones which avoided the problem of anomalies. The type I superstring with Chan-Paton gauge symmetry group SO(32) was also anomaly-free, but there the list ended. The only consistent string theories which contained gauge groups in ten dimensions were the $E_8 \times E_8$ and spin(32)/$Z_2$ heterotic strings and the SO(32) type I superstring.

The type II superstring was still a fully consistent string theory, but it had no gauge group in ten dimensions. It had also been demonstrated[15] that the chiral type II superstring was also free of gravitational anomalies, so in this respect it was on an equal footing with the SO(32) type I and heterotic strings. However, it appeared that there was no way of introducing a gauge group into the theory and that it was hopeless for phenomenology. It had been proved for Kaluza-Klein theories, using index theorems, that one cannot obtain chiral fermions in 4 dimensions starting from a 4+n dimensional theory unless the higher dimensional theory already contained gauge fields.[16] So as far as Kaluza-Klein theories are concerned, gauge fields must be in the original theory - they cannot simply be introduced via the compactification. Now type II theories are string theories, so it may be that something inherently string-like allows one to get around this theorem. But the fact still remains that type II superstrings have no gauge groups in ten

dimensions, and the only ways that were known in 1985 of introducing symmetry into string theories did not work with type II strings. The Chan-Paton method attaches charges onto the ends of open strings, but type II strings are closed strings, so one can only obtain charge singlets. The FKS construction introduces gauge symmetry via the compactification of the left-moving bosonic side of the heterotic string, but the type II superstring is supersymmetric on both sides.

There were, in addition to this, two no-go theorems circulating in 1985 which had as their conclusions that it was impossible to incorporate nonabelian gauge groups into type II superstring theories via compactification without breaking supersymmetry and generating large fermion masses. The first of these by Friedan, Qiu, and Shenker[17] approached the problem from the point of view of superconformally invariant two-dimensional field theories.[18-19] Here the type II superstring gets broken up into Ramond (fermionic) and Neveu-Schwarz (bosonic) sectors of a 2-dimensional conformal field theory, and the supersymmetry is a symmetry of the 2-dimensional space (the string worldsheet) rather than the 10-dimensional embedding space. The superconformal algebra generated by the type II superstring has a central charge $\hat{c}=10$, and a compactification of the string theory corresponds to replacing a chunk of the theory with $\hat{c}=6$ by some other nontrivial superconformal theory that also has $\hat{c}=6$. The original theory consists of ten scalar fields (corresponding to the spacetime dimensions) and their supersymmetric partners, which contribute a total $\hat{c}=10$; whereas the "compactified" string theory has 4 scalar fields and 4 fermions for $\hat{c}=4$, but there is in addition some other superconformal theory (or theories) contributing the remaining $\hat{c}=6$. These constitute the most general compactifications of type II superstrings. What Friedan, Qiu, and Shenker did

was to consider the most basic features of all of these theories, namely their super-Virasoro and Kac-Moody algebras, and they claimed that the existence of massless fermions (and hence unbroken supersymmetry) was incompatible with having a nonabelian gauge algebra. Their conclusion was that nonabelian gauge symmetry could not be introduced into type II models via compactification without destroying all hope of obtaining phenomenologically viable models.

A second no-go theorem was put forth by Antoniadis, Bachas, Kounnas, and Windy which reached the same conclusion.[20] They too proposed the idea of considering compactified strings as four-dimensional objects (that is as superconformal field theories with scalar fields and fermions contributing $\hat{c}=4$) plus some additional conformal fields (contributing the remaining $\hat{c}=6$). In their case the additional fields consisted of free fermions, and they showed, following the work of Goddard and Olive,[21] that these free fermions could be used to represent some interesting nonabelian gauge groups. However, they showed as well that these nonabelian groups generated fermion masses and thus could not be used in a realistic string model. Their proof relied on the fact that the free fermions transformed in the adjoint representation of a subgroup H of G, the nonabelian group, and that G/H had to be a symmetric space. This later fact was shown to be inconsistent with having a massless Ramond sector, and they too concluded that type II strings could not be compactified in a way that introduces a nonabelian gauge group.

Since there was no way to introduce gauge symmetry into type II theories by way of compactification, nor directly into the ten-dimensional theory à la Chan-Paton, it appeared that, despite the fact that they were consistent, one-loop finite, and anomaly-free, type II superstrings could not be made a phenomenologically viable theory. In fact, in early 1987, when the first textbook on superstrings came out - the two-volume set by Green, Schwarz, and

Witten[4] - this view of type II strings was apparent. In volume I, where the type II superstring is first introduced, the authors conclude: "It has no freedom to introduce a Yang-Mills group."[22]

Heterotic strings already had a gauge group in ten dimensions - the problem then became that of compactifying the extra six dimensions. By asking what conditions had to be imposed to bring the $E_8 \times E_8$ heterotic string from 10 down to 4 dimensions while maintaining N=1 supersymmetry, Candelas, Horowitz, Strominger, and Witten[23] had discovered that compactification on Calabi-Yau manifolds worked well. A Calabi-Yau manifold is a Ricci-flat Kähler manifold with three complex dimensions, SU(3) holonomy, and vanishing first Chern class. What Candelas et al. derived was a field theory limit in four dimensions with $E_6$ as a grand unifying group, N=1 supersymmetry and the possibility of obtaining chiral fermions, and a whole host of possibilities for constructing a unified theory of all the fundamental interactions. The problem then becomes that of finding the right Calabi-Yau manifold out of thousands of possibilities and of breaking the gauge symmetry down to SU(3)×SU(2)×U(1) while obtaining three families of chiral fermions with the right Yukawa couplings. The final word is not yet in on this approach - it certainly has some excellent possibilities.

In principle, the Calabi-Yau compactification scheme could be used directly on the ten-dimensional heterotic string to obtain a string theory in four dimensions rather than just a field theory limit. But then one must face the problem of describing string propagation on a complicated background. Candelas et al. found that all the restrictions that arise from requiring that the four-dimensional field theory has N=1 supersymmetry have counterparts in terms of the string theory. However, a workable approach to dealing with all

the inherently string-like aspects of Calabi-Yau compactifications does not exist at present.

The situation then, as it stood about two years ago, for reducing string theories down to four dimensions and obtaining something manageable (and still a string theory - not a field theory limit) was that one could use either orbifolds[24] or simple $U(1)^6$ toroidal compactifications[25] to compactify the extra dimensions. None of these techniques have the simple economy of the FKS method, though. This consists of a nontrivial toroidal compactification, in which a torus is formed by putting the momenta on the root lattice of a Lie group. It is relatively straightforward to deal with - plus it accomplishes two things at once: it compactifies the spacetime at the same time that it introduces gauge symmetry into the problem. However, as stated before, the FKS method only works on bosonic strings. What was desired was a modification of the FKS method that would allow it to be applied to ten-dimensional superstrings. By starting with a ten-dimensional string and reducing it to four dimensions, one would hope to obtain rank 6 groups, which would fit the standard model a lot tighter than the rank 16 groups of the heterotic string. Lastly, what was desired most was simplicity. Heterotic strings compactified on Calabi-Yau manifolds or orbifolds require two different internal spaces working in conjunction. Would it not be preferable to start in ten dimensions and use only the simplest compactification, and at the same time introduce a gauge group of low rank? Such a string theory in four dimensions would be more economical than any of the other models, which seem to provide too much freedom and are overly complicated.

A first step in this direction was taken in ref. [1]. Here a new affine Kac-Moody construction in terms of Neveu-Schwarz operators was used to

compactify the Neveu-Schwarz bosonic string. The "old" superstring, or Neveu-Schwarz-Ramond (NSR) spinning string, consists of two theories: the NS[26] which is bosonic, and the R[27] which is fermionic. The "new" Green-Schwarz spacetime supersymmetric string, the superstring,[28] is obtained from the old NSR string by projecting out certain of the NS and R states. The resulting spectrum, as shown by Gliozzi, Scherk, and Olive (GSO),[29] is spacetime supersymmetric - and the projections are known as GSO projections. The NS string has a tachyon in its spectrum, but this can be eliminated by making a GSO projection. Both the NS and R strings are constructed in terms of worldsheet (two-dimensional) fermion operators as well as two-dimensional scalars (the string coordinates, of which there are ten). The resulting theory is supersymmetric on the worldsheet. This is the main difference between the NSR string and the superstring - they are both equivalent formulations of the same theory, but the NSR string is explicitly supersymmetric on the two-dimensional worldsheet, whereas the superstring is explicitly spacetime supersymmetric in ten dimensions.

In ref. [1], a hybrid closed string consisting of a right-moving superstring and a left-moving NS bosonic string was constructed, and using the new algebraic compactification we were able to introduce the gauge group $SU(2)^6$ into the theory. The result is a four-dimensional, Lorentz-invariant, tachyon-free theory that at one loop is invariant under a subgroup of the modular group. In four dimensions, the massless sector consists of N=4 supergravity coupled to N=4 supersymmetric Yang-Mills theory with the gauge group $SU(2)^6$. We thus found an analog of the FKS construction, which could be used on Neveu-Schwarz strings.

The corresponding Ramond sector was considered in ref. [2]. By adding a Ramond sector on the left crossed into the superstring on the right (R×SS) to

the theory we already had, namely the NS crossed with the superstring (NS×SS), we obtained the type II superstring in ten dimensions in a hybrid formulation (NSR×SS) - the old formulation on the left, the new on the right. Our four-dimensional theory was thus a compactification of the type II superstring - and one that introduced a nonabelian gauge group $SU(2)^6$. Actually, we did not really have to add in the Ramond sector; it was already there. The modular subgroup invariant amplitude of the NS string alone equals the fully modular invariant amplitude of the NS plus R strings together. In this way we were able to demonstrate that the compactified type II string was fully one-loop modular invariant.

So what about the no-go theorems that said it was impossible to generate nonabelian gauge groups by compactifying type II superstrings? These theorems were not addressed in refs. [1,2] because they had no bearing on the way our results were derived. We may still ask, though, why the no-go theorems do not apply to the new model (as well as its fermionic generalizations). While the details of these theorems still hold, the overall conclusions reached from them were arrived at too hastily. The no-go theorems were circumvented because the new compactifications of the type II superstring work differently on the left- and on the right-moving sides - that is they are asymmetric compactifications. The no-go theorems claimed that introducing nonabelian symmetry would generate massive fermions and would break supersymmetry. This is true enough, but it is not enough to rule out type II superstrings altogether. Type II strings are N=2 supersymmetric in ten dimensions. Using a $U(1)^6$ toroidal compactification generates an N=8 supersymmetric string in four dimensions. The new models, however, use a $U(1)^6$ toroidal compactification only on the right-moving side, the superstring side, whereas the NS construction is used on the left side. The fermionic, or Ramond,

sector on the left side does become massive, and the spacetime supersymmetry is lost on the left side. But we still have a fully supersymmetric theory where the supersymmetry is carried by the right-movers alone, and in four dimensions we get N=4 supersymmetry, instead of N=8.

The main result of ref. [2], however, was that we were able to generalize this method of incorporating nonabelian symmetry into compactified string theories. By replacing the six compactified dimensions of the type II superstring by free fermions, we were able to introduce any dimension 18 semi-simple Lie group into the theory - so in addition to $SU(2)^6$ we could have $SU(4)\times SU(2)$ or $SO(5)\times SU(3)$. The latter two gauge groups are of rank 4, which exactly matches the rank of the standard model. This is a major departure from the FKS system, in which the number of compactified dimensions always equals the rank of the gauge group. Here we obtain gauge groups which are of dimension three times the number of compactified coordinates ($18=3\times6$).

From the point of view of superconformal field theories, these compactifications can be viewed as four-dimensional models because the $\hat{c}=6$ part of the theory has been replaced by something other than string coordinates - namely 18 free fermions, each contributing $\hat{c}=\frac{1}{3}$. But these string theories still possess the same number of degrees of freedom as the ten-dimensional theories because the total super-Virasoro central charge is still $\hat{c}=10$. Nonetheless, we will refer to them as "four-dimensional" strings.

In the past year or two, four-dimensional strings have become one of the more active fields of research within string theory. In a paper by Narain,[30] it was shown that the lattice FKS compactification used in the heterotic string can be generalized to include even, self-dual, Lorentzian lattices, and these can be used to compactify the heterotic string all the way down to four dimensions. The resulting models are N=4 supersymmetric and

carry rank 22 gauge algebras. Their physical interpretation is that by combining the left and right momenta to give rise to an even, self-dual, Lorentzian lattice, one is in effect switching on Wilson lines and anti-symmetric tensor fields on the toroidally compactified string theory.[31] A whole new class of orbifolds, called asymmetric orbifolds,[32] was developed as well. These are theories in which the left-moving and right-moving degrees of freedom live on different orbifolds. The starting point for these theories is the Narain compactifications. It is these generalized lattices that are rotated and shifted, but in such a way as to not mix up the left- and right-moving Hilbert spaces. It was shown by Dixon, Kaplunovsky, and Vafa[33] that the models in ref. [2] can be reinterpreted as asymmetric orbifolds. Furthermore, by twisting these models they found examples of N=1 theories with chiral fermions based on the type II superstring. Two groups of collaborators[34-35] have developed an elaborate but very general formalism for writing down modular-invariant, four-dimensional models. Their work was based primarily on the heterotic string, but the techniques they developed can be applied to type II strings and can incorporate our new symmetry groups as well.[36]

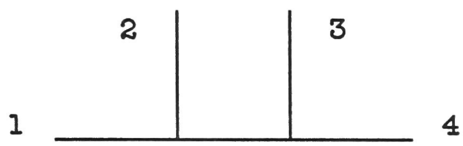
In ref. [3] we developed a diagrammatic representation of this model-building formalism. We simplified the approach and showed how the constraints on constructing models which appear to satisfy higher-loop modular invariance[37] in fact follow from the requirements of unitarity at the tree and one-loop levels. Since the real guide we have in the construction of string theories at present is the requirement of unitarity, it is useful to have a discussion based directly on it. The formalism can then be used to investigate twisted models, which break the N=4 supersymmetry and produce

chiral fermions in four dimensions. Also, our formalism applies equally well to heterotic strings as well as type II superstrings.

The following five sections are organized as follows: sections I and III contain background information on the operator formalism and on superconformal algebras. Sections II, IV, and V contain original work and are comprised, respectively, of refs. [1], [2], and [3].

## I. The Operator Formalism

String theory comes in many guises. The original formalism which grew out of the old dual resonance theory is called the operator formalism. This is a perturbative approach in which propagators and vertex operators are used to constuact "diagrams" in the interacting theory. To lowest order one can construct tree diagrams.




These are the diagrams which correspond to the original Veneziano amplitude. Veneziano wrote down his amplitude as a guess of the form an amplitude should have to display the properties of duality, crossing symmetry, and Regge trajectories:[38]

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \tag{I.1}$$

where
$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \tag{I.2}$$

is the Euler gamma function, and $\alpha(s)$ is the Regge trajectory, having the form:

$$\alpha(s) = \alpha(0) + \alpha' s. \tag{I.3}$$

s,t, and u are the Mandelstam variables

$$\begin{aligned} s &= -(p_1 + p_2)^2 \\ t &= -(p_2 + p_3)^2 \\ u &= -(p_1 + p_3)^2. \end{aligned} \tag{I.4}$$

The metric we are using is $\{-++\cdots+\}$ and $m^2 = -p^2$. The Mandelstam variables obey the identity

$$s + t + u = \sum m_i^2. \tag{I.5}$$

The Veneziano amplitude, eq. (I.1), is explicitly symmetric in the s and t channels. The parameters $\alpha(0)$ and $\alpha'$ are known respectively as the Regge intercept and the Regge slope.

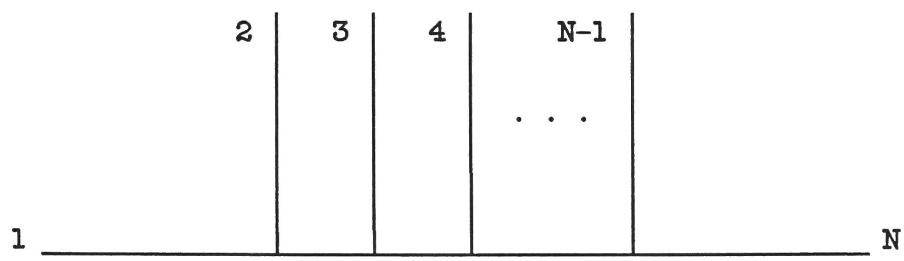
Nambu showed that the Veneziano amplitude was the solution of a relativistic string equation.[39] He wrote down an action as the area of the worldsheet traced out as a string propagates through time. Then, following the work of Goddard, Goldstone, Rebbi, and Thorn (GGRT),[40] it was shown that everything previously known about the factorized Veneziano model could be derived from a first quantized Lagrangian describing a free string. They found that it was possible to quantize the theory in two different ways - one which is explicitly covariant but where the absence of ghosts must be proved (called the covaiant gauge), the other where only physical oscillators appear but where Lorentz invariance has to be proved (called the light-cone gauge).

There are several other formulations of string theory as well. There is the Polyakov approach[41] in terms of path integrals as well as a second-quantized string field theory.[6] String theory can also be formulated purely in terms of two-dimensional conformal field theory.[18-19] Most of the work

presented in this thesis will be presented in the operator formalism in light-cone gauge. This is the most physical picture. In some sections, however, I will use the covariant picture, and much of the results on the new representation of super-Virasoro and super-Kac-Moody generators in terms of free fermions will be presented in conformal field theory language. In section IV, I will discuss the N=2 superconformal algebra and an attempt that I made to incorporate this into an extended super-Kac-Moody algebra.

String Perturbation Theory

Once the vertex operators V(k) and propagators $\Delta$ were identified in the factorized N-point amplitudes, it became a staightforward problem to calculate string amplitudes. The tree diagrams can be represented as in the following figure:

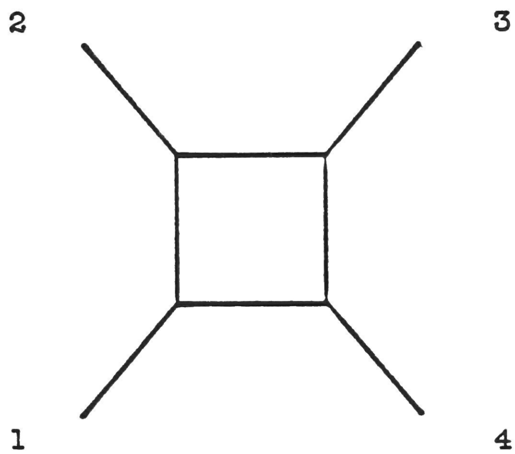



The amplitude then is

$$A_{tree}(12\cdots N) = \langle 1|V(2)\Delta V(3)\cdots\Delta V(N-1)|N\rangle \tag{I.6}$$

and the problem of calculating amplitudes becomes an algebraic one involving the commutation relations of the oscillators in the definitions of $V(k)$ and $\Delta$.

The next order in the perturbative expansion is the one-loop level. The one-loop amplitude is obtained in an ad hoc manner by sewing together a tree diagram.




$$A_{loop}(1234) = \int dp \; tr(\Delta V(1)\Delta V(2)\Delta V(3)\Delta V(4)) \qquad (I.7)$$

The integration is over the loop momentum and the trace is over all the intermediate states. For closed strings, the loop propagating around on itself is topologically a torus, and the amplitude is constrained in such a way as to properly take into account the symmetries of the torus. This involves the idea of modular invariance, which will be defined later. One can go on to calculate higher-loop amplitudes in the operator formalism, but the problem quickly becomes unmanageable. The two-loop amplitude must display all the symmetries of a double donut. Here one is better off using the path

integral formulation, where (at least in theory) one can calculate any higher-loop diagram.

In section V, I will discuss the connection between one-loop modular invariance and higher-loop modular invariance. There it will be shown that the requirements of spacetime factorization for the vertices and one-loop modular invariance guarantee higher-loop modular invariance in the context of models built out of free fermion fields.[3] A discussion of the connection between modular invariance and unitarity will also be presented in section II.

## The Superstring

The Green-Schwarz superstring in the light-cone gauge can be derived from the supersymmetrical action[7]

$$S = \int d\sigma d\tau \, \left(\frac{-1}{4\pi\alpha'} \partial_\alpha X^i \partial^\alpha X^i + \frac{i}{4\pi} \bar{S}\gamma^- \zeta^\alpha \partial_\alpha S\right). \qquad (I.8)$$

The $X^i(\sigma,\tau)$ are the transverse coordinates ($i=1,\ldots,8$ in 10 dimensions) and the $S^{Aa}(\sigma,\tau)$ are their fermionic partners. $A=1,2$ are two-dimensional spinor coordinates, while $a=1,\ldots,32$ are the spinor coordinates in ten-dimensional spacetime. The requirements that S be Majorana and Weyl reduce the 64 complex components of $S^{Aa}$ to 32 real components. The Weyl condition is

$$h_A^{ab} S^{Ab} = 0 \qquad \text{A not summed} \qquad (I.9)$$

where $h_A^{ab}$ denotes the Weyl projections $\frac{1}{2}(1\pm\gamma_{11})$ and $(\gamma^\mu)^{ab}$ are spacetime Dirac matrices in a Majorana representation.

The light-cone condition for the string coordinates

$$X^+(\sigma,\tau) = x^+ + 2\alpha' p\tau \qquad (1.10)$$

as well as the constraint equations coming from the reparametrization invariance of the action

$$\partial_\sigma X\cdot\partial_\tau X = 0 \qquad (1.11a)$$

$$\partial_\sigma X\cdot\partial_\sigma X + \partial_\tau X\cdot\partial_\tau X = 0 \qquad (1.11b)$$

are used to eliminate the dependent variables $X^+$, $X^-$ in terms of the physical light-cone coordinates $X^i$.

Similarly, for the fermionic partners a light-cone condition is imposed

$$(\gamma^+)^{ab}S^{Aa} = 0 \qquad (I.12)$$

which further restricts $S^{Aa}$ to 16 real components. The Dirac equation then reduces the number of independent $S^{Aa}$ components to 8, which then matches the number of bosonic degrees of freedom for the $X^i$. $\bar{S}$ is given by

$$\bar{S}^{Aa} = S^{\dagger Bb}(\gamma^o)^{ba}(\zeta^o)^{BA} \qquad (I.13)$$

$$(\zeta^\alpha)^{BA} = \text{worldsheet Dirac matrices,}$$

and the action is invariant under the supersymmetry transformations

$$\delta X^i = (p^+)^{-1/2}\,\bar{\epsilon}\gamma^i S$$

$$\delta S = i(p^+)^{-1/2}\,\gamma_-\gamma_\mu(\zeta\cdot\partial X^\mu)\epsilon. \qquad (I.14)$$

The parameters $\epsilon^{Aa}$ are Majorana-Weyl spinors in 10 dimensions.

The equation of motion and boundary conditions for the $X^i$ variables are given by

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)X^i(\sigma,\tau) = 0 \tag{I.15}$$

and

$$X^i(0,\tau) = X^i(\pi,\tau), \tag{I.16}$$

the latter being for closed strings (the parameter $\sigma$ is normalized to take the values 0 to $\pi$); whereas for open strings the boundary conditions are

$$\left.\frac{\partial}{\partial\sigma} X^i\right|_{\sigma=0} = \left.\frac{\partial}{\partial\sigma} X^i\right|_{\sigma=\pi}. \tag{I.17}$$

The S variables satisfy the two-dimensional Dirac equation, which in terms of its components may be written

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma}\right)S^{1a} = 0 \tag{I.18}$$

$$\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma}\right)S^{2a} = 0. \tag{I.19}$$

For open superstrings the boundary conditions are

$$S^{1a}(0,\tau) = S^{2a}(0,\tau) \tag{I.20a}$$

$$S^{1a}(\pi,\tau) = S^{2a}(\pi,\tau). \tag{I.20b}$$

This requires that $S^1$ and $S^2$ both must be expanded in terms of a single set of oscillators $S^a_n$

$$S^{1a} = \sum_n S^a_n e^{-in(\tau-\sigma)} \tag{I.21a}$$

$$S^{2a} = \sum_n S^a_n e^{-in(\tau+\sigma)}. \tag{I.21b}$$

The closed-string boundary conditions are

$$S^{Aa}(0,\tau) = S^{Aa}(\pi,\tau) \tag{I.22}$$

which lead to expansions in terms of two independent oscillators (right-movers and left-movers)

$$\begin{aligned} S^{1a} &= \sum_n S^a_n e^{-2in(\tau-\sigma)} \\ S^{2a} &= \sum_n \tilde{S}^a_n e^{-2in(\tau+\sigma)}. \end{aligned} \tag{I.23}$$

For $X^i$ with closed-string boundary conditions, the normal-mode expansion is

$$X^i(\sigma,\tau) = x^i + p^i\tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}(\alpha^i_n e^{-2in(\tau-\sigma)} + \tilde{\alpha}^i_n e^{-2in(\tau+\sigma)}). \tag{I.24}$$

Canonical quantization gives

$$\begin{aligned} [\alpha^i_n,\alpha^j_m] = [\tilde{\alpha}^i_n,\tilde{\alpha}^j_m] &= n\delta_{n,-m}\delta^{ij} \\ [\alpha^i_n,\tilde{\alpha}^j_m] &= 0 \\ \{S^a_m,\bar{S}^b_n\} &= (\gamma^+h)^{ab}\delta_{m,-n} \\ [\alpha^i_m,S^a_n] &= 0 \\ \{S^a_m,\tilde{S}^b_n\} &= 0. \end{aligned} \tag{I.25}$$

The physical on-mass-shell states are given by

$$\frac{1}{4}\alpha' m^2 = N \tag{I.26}$$

where $$N = \sum_{n=1}^{\infty} (\alpha^i_{-n}\alpha^i_n + \tfrac{n}{2}\bar{S}_{-n}\gamma^- S_n), \tag{I.27}$$

and the left-right constraint is given by $N=\tilde{N}$, where $\tilde{N}$ is as in eq. (I.27), but with $\alpha^i_n$ and $S_n$ replaced by $\tilde{\alpha}^i_n$ and $\tilde{S}_n$. This constraint says that the energy is carried equally by the left-movers and right-movers.

The NSR String

The Green-Schwarz formalism is the "new" superstring formalism which is explicitly spacetime supersymmetric. The "old" NSR formalism[26-27] is explicitly supersymmetric on the worldsheet and is given in terms of worldsheet fermions $b^i$ and $d^i$ rather than the spacetime fermions $S^a$. The string coordinates $X^i$ are the same as before, but now they are treated as worldsheet scalars, and their fermionic partners are either NS b-operators, which are anti-periodic in the closed string theory, or R d-operators, which are periodic. The NS oscillators are half-integrally moded and in the quantized theory obey

$$\begin{aligned} \{b^i_r, b^j_s\} &= \delta_{r,-s}\delta^{ij} \\ b^i(0,\tau) &= -b^i(\pi,\tau), \end{aligned} \tag{I.28}$$

while the R oscillators are integrally moded:

$$\begin{aligned} \{d^i_n, d^j_m\} &= \delta_{n,-m}\delta^{ij} \\ d^i(0,\tau) &= d^i(\pi,\tau). \end{aligned} \tag{I.29}$$

In the ten-dimensional spacetime, the b operators are bosonic, while the d operators are fermionic. This may be seen from the fact that the d operators have zero modes which form a Clifford algebra.

For the bosonic sector the mass operator is

$$\alpha' m^2 = N = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=1/2}^{\infty} r b^i_{-r} b^i_r - \frac{1}{2} . \tag{I.30}$$

Thus the NS theory has a tachyon in its spectrum, and its massless states are vector states $b^i_{-\frac{1}{2}}|0\rangle$. The lowest level of the R sector is at the massless level and is a spinor state.

The supersymmetric spectrum of the superstring is regained when the NS and R theories are combined into one theory and the GSO-projections are made on the states. These eliminate the tachyon from the spectrum, impose Majorana-Weyl conditions on the fermionic states, and lead to a spacetime supersymmetric theory.

In both the Green-Schwarz and the NSR formalism in light-cone gauge, the Lorentz invariance of the theory must be checked explicitly. In the canonical formulation the Lorentz generators can be found in terms of the oscillators of the theory - and the index i that counts the number of spacetime dimensions is treated as a free parameter. It is found that for the superstring to be Lorentz invariant, the number of dimensions must equal ten.

Interactions and Modular Invariance

The simplest interacting theory is the open Veneziano model at the tree level. The propagator is written as[7]

$$\Delta = \alpha' \int_0^1 x^{N-2} dx \tag{I.31}$$

and the vertex operator for tachyon emission is

$$V_0(k) = g \ :e^{ik\cdot X}: \ . \tag{I.32}$$

The 4-point tree amplitude

$$A(1234) = \langle 0,p_1 | V(p_2)\Delta V(p_3) | 0,p_4\rangle \tag{I.33}$$

reproduces the Veneziano amplitude, eq. (I.1).

For closed strings, the σ-dependence of the string coordinates can be incorporated into the propagators, which is then given by

$$\Delta_c = \frac{\alpha'}{2\pi} \int d^2z \ |z|^{\frac{p^2}{4} - 2} z^N \bar{z}^{\tilde{N}} \ . \tag{I.34}$$

At tree level, the integration is over the entire complex plane, while at one-loop the integration region is fixed from the requirements of modular invariance.

The one-loop, 4-point, closed string amplitude is calculated as

$$A_{loop}(1234) = \int dp\ \mathrm{tr}(\Delta_c W(1)\Delta_c W(2)\Delta_c W(3)\Delta_c W(4)) \qquad (I.35)$$

where
$$W(k) = :e^{ik\cdot X}:$$

and X is as given in eq. (I.24). The problem then becomes that of tracing over all the intermediate states and performing the loop integration. The final answer can be written in terms of Jacobi theta functions when the dummy variables $z_1,\ldots,z_4$ from the propagators are replaced by[7]

$$\nu_I = \frac{\ln z_1 z_2 \cdots z_I}{2\pi i} \qquad I = 1,2,3$$
$$\tau = \nu_4 = \frac{\ln w}{2\pi i}$$
$$w = z_1 z_2 z_3 z_4. \qquad (I.36)$$

In terms of these the loop amplitude can be written in the form

$$A_{loop}(1234) = \int (\prod_{I=1}^{3} d^2\nu_I)\ d^2\tau\ (\mathrm{Im}\ \tau)^{-2} A(\nu_I,\tau). \qquad (I.37)$$

The amplitude $A(\nu_I,\tau)$ involves the doubly-periodic Jacobi theta functions and can be shown to be invariant under

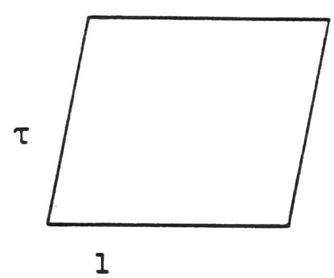
$$\nu_I \to \nu_I + 1$$

$$\nu_I \to \nu_I + \tau. \tag{I.38}$$

This suggests that to avoid overcounting we restrict the $\nu_I$ integration to the region

$$\begin{gathered} 0 \le \mathrm{Im}\ \nu_I \le \mathrm{Im}\ \tau \\ -\tfrac{1}{2} \le \mathrm{Re}\ \nu_I \le \tfrac{1}{2}. \end{gathered} \tag{I.39}$$

Since a closed string one-loop diagram is topologically a torus, we should expect to see the symmetries of the torus reflected in the one-loop amplitude. This is what we observe in the $\nu_I$ variables in eq. (I.39) - they are seen to live on a space which can be drawn as a parallelogram on which




the tops and sides are identified so as to form a torus. The restriction to the region given by eq. (I.39) ensures that we only integrate over each torus once.

The shape of the torus can be seen to be given by the complex parameter $\tau$. Since we only want to integrate over each torus once, we must restrict the $\tau$ integration to a fundamental region corresponding to inequivalent tori only. The parallelogram drawn above can be thought of as the unit cell of a lattice having 1 and $\tau$ as its basis vectors. Then, the same lattice which is spanned by 1 and $\tau$ is also spanned by a new set of basis vectors where the new set is obtained from the old by a modular transformation[42-43]

$$\tau \to \frac{a\tau + b}{c\tau + d} \tag{I.40}$$

with integers a,b,c,d such that ad - bc = 1. (The other basis vector 1 is transformed to a new vector 1′ as well, but the reparametrization invariance is used to rescale it back to 1 again.) Thus any two tori connected by a modular transformation are equivalent, and we will be integrating over the same torus an infinite number of times unless we restrict the integration to the fundamental region of the modular group

$$\begin{aligned} F: \quad -\frac{1}{2} < \operatorname{Re}\tau < \frac{1}{2} \\ \operatorname{Im}\tau > 0 \\ |\tau| > 1. \end{aligned} \tag{I.41}$$

It can be shown that the integrand $A(\nu_I, \tau)$ in eq. (I.37) is invariant under the modular transformations, eq. (I.40), and with the integration region restricted to the fundamental region F we obtain a well-defined result,[44] which in the case of superstrings will be finite up to the correct poles and cuts in agreement with unitarity.

<u>The FKS Construction</u>

The Frenkel-Kac-Segal (FKS) construction[12-14] provides a natural way of introducing internal symmetry into bosonic string theories which has been successfully exploited in the heterotic string to provide string theories with phenomenologically viable gauge groups. In the heterotic string the symmetry is incorporated by using the FKS construction for 16 of the left-moving 26-dimensional bosonic degrees of a closed string while the right-moving degrees of freedom are those of the 10-dimensional superstring.

In the FKS construction the string is compactified by quantizing the internal momenta on the even root lattice of a simply-laced Lie group G. The vertex operators $V_o(r,z)$ for the tachyon states are used to construct the affine algebra $\hat{G}$:

$$V_o(r,z) = :e^{2ir\cdot X(z)}: \qquad \text{(I.42)}$$
$$z = e^{2i(\tau + \sigma)} \quad \text{(left-movers)}.$$

The affine $\hat{G}$ generators are the $\alpha_n^I$ and $X_n(r)$, where $r^I$ = the roots of G and

$$X_n(r) = \frac{c_r}{2\pi i} \oint \frac{dz}{z} z^n \, V_o(r,z). \qquad \text{(I.43)}$$

The $c_r$ are cocycle operators which are functions of the momentum. These affine generators commute with the Lorentz generators, and the compactified spectrum falls into representations of the Kac-Moody algebra. The rank of the gauge group obtained equals the number of compactified dimensions. For the heterotic string, the modular invariance of the interacting theory requires

that the root lattice used in the compactification be that of an even, self-dual, and simply-laced group. Such lattices only exist in multiples of 8 dimensions, and for 16 dimensions the only possibilities are for the groups $E_8 \times E_8$ and spin(32)/$Z_2$.

In the next section this construction in light-cone gauge will be generalized so as to work for NS vertex operators. A fully modular invariant model in four dimensions is then constructed, which corresponds to the first consistent compactification of type II superstrings involving a nonabelian group.

## II. The $SU(2)^6$ Model

In this section a new string model is presented. In ten dimensions, it is a hybrid version of the type II superstring. An algebraic compactification involving a new affine Kac-Moody constuction in terms of Neveu-Schwarz operators is used to reduce the theory to four dimensions while incorporating the symmetry group $SU(2)^6$ into the model. The resulting theory is Lorentz-invariant, tachyon-free, and fully modular invariant; the field content of the massless sector is N=4 supergravity coupled to N=4 supersymmetric Yang-Mills theory with the gauge group $SU(2)^6$. The work in this section is contained in ref. [1] and part of ref. [2].

### A Hybrid Type II Superstring

The model in ten dimensions consists of a closed orientable string whose right-moving and left-moving degrees of freedom are the operators of the Green-Schwarz superstring and the NSR spinning string, respectively. For the moment, let us examine just the NS sector on the left side. This alone is a bosonic string, but coupled to the superstring on the right it becomes N=1 supersymmetric in 10 dimensions and N=4 in four dimensions. I will denote this model as the NS×SS string. Later, we will look at the Ramond string crossed into the superstring - R×SS.

In light-cone gauge and in 4 dimensions, the transverse degrees of freedom of the NS×SS theory are:

$$X^{\hat{\imath}}(\tau-\sigma) = \frac{x^{\hat{\imath}}}{2} + \alpha' p^{\hat{\imath}}(\tau-\sigma) + \frac{i\sqrt{2\alpha'}}{2} \sum_{n\neq 0} \frac{1}{n}\, \alpha^{\hat{\imath}} e^{-2in(\tau-\sigma)} \tag{II.1a}$$

$$X^{I}(\tau-\sigma) = \bar{x}^{I} + 2\alpha' \bar{p}^{I}(\tau-\sigma) + \frac{i\sqrt{2\alpha'}}{2} \sum_{n\neq 0} \frac{1}{n}\, \alpha^{I} e^{-2in(\tau-\sigma)} \tag{II.1b}$$

$$S^{a}(\tau-\sigma) = \sum_{n=-\infty}^{\infty} S^{a}_{n}\, e^{-2in(\tau-\sigma)} \tag{II.1c}$$

$$\tilde{X}^{\hat{\imath}}(\tau+\sigma) = \frac{x^{\hat{\imath}}}{2} + \alpha' p^{\hat{\imath}}(\tau+\sigma) + \frac{i\sqrt{2\alpha'}}{2} \sum_{n\neq 0} \frac{1}{n}\, \tilde{\alpha}^{\hat{\imath}} e^{-2in(\tau+\sigma)} \tag{II.2a}$$

$$\tilde{X}^{I}(\tau+\sigma) = \tilde{x}^{I} + 2\alpha' \tilde{p}^{I}(\tau+\sigma) + \frac{i\sqrt{2\alpha'}}{2} \sum_{n\neq 0} \frac{1}{n}\, \tilde{\alpha}^{I} e^{-2in(\tau+\sigma)} \tag{II.2b}$$

$$\lambda^{i}(\tau+\sigma) = \sum_{s=-\infty}^{\infty} b^{i}_{s}\, e^{-2is(\tau+\sigma)}. \tag{II.2c}$$

Here

$$a = 1,\ldots,32 \qquad i = 1,\ldots,8$$

$$\hat{\imath} = 1,2 \qquad I = 1,\ldots,6$$

$$\mu = 0,1,\ldots,9$$

$$n \in Z$$

$$s \in Z + \tfrac{1}{2} \tag{II.3}$$

so that $\hat{\imath}$ denotes the 4-dimensional spacetime coordinates, while I labels the compactified coordinates. Tildes are used to distinguish the left-moving string coordinates.

Light-cone quantization gives the following nonzero commutation relations for these physical modes:

$$[\alpha^{i}_{n}, \alpha_{m}{}^{j}] = [\tilde{\alpha}^{i}_{n}, \tilde{\alpha}^{j}_{m}] = n\delta_{n,-m}\delta^{ij}$$

$$\{b^{i}_{s}, b^{j}_{r}\} = \delta_{s,-r}\delta^{ij}$$

$$\{S_n^a, \bar{S}_m^b\} = (\gamma^+ h)^{ab}\delta_{m,-n} \tag{II.4}$$

$$[x^{\hat{i}}, p^{\hat{j}}] = i\delta^{\hat{i}\hat{j}}$$

$$[\tilde{x}^I, \tilde{p}^J] = [\bar{x}^I, \bar{p}^J] = \frac{i}{2}\delta^{IJ}. \tag{II.5}$$

In ten dimensions the fermions are Majorana-Weyl spinors

$$h^{ab}S_n^b = 0$$

$$(\gamma^+)^{ab}S_n^b = 0$$

$$h^{ab} = \frac{1}{2}(1+\gamma_{11})^{ab}$$

$$\bar{S}_n^a = S_n^b(\gamma^0)^{ba}. \tag{II.6}$$

$(\gamma^\mu)^{ab}$ are imaginary in the Majorana representation, $\gamma^0$ is anti-symmetric, $\gamma^i$ and $\gamma^9$ are symmetric, $\gamma_{11} = \gamma^0\gamma^1 \cdots \gamma^9$ is real symmetric. The half-integrally moded anti-commuting operators $b_s^i$ describe bosons. The coordinate conditions of light-cone gauge give the dependent operators and a constraint equation

$$X^+(\sigma,\tau) = X^+(\sigma+\tau) + \tilde{X}^+(\sigma-\tau) = x^+ + 2\alpha' p\tau$$

$$X^-(\sigma,\tau) = X^-(\sigma+\tau) + \tilde{X}^-(\sigma-\tau)$$

$$= 2\alpha' p^-\tau + \frac{\sqrt{2\alpha'}}{2}\sum_{n\neq 0}\left[\alpha_n^- e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^- e^{-2in(\tau+\sigma)}\right] \tag{II.7}$$

where for $n\neq 0$,

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}p^+}L_n \qquad \tilde{\alpha}_n^- = \frac{1}{\sqrt{2\alpha'}p^+}\tilde{L}_n \tag{II.8}$$

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty}\left[\alpha_{n-m}^i\alpha_m^i + \frac{1}{2}(m-\frac{n}{2})\bar{S}_{n-m}\gamma^- S_m\right] \tag{II.9a}$$

$$\tilde{L}_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \tilde{\alpha}^i_{n-m}\tilde{\alpha}^i_m + \frac{1}{2} \sum_{s=-\infty}^{\infty} (s-\frac{n}{2}) b^i_{n-s} b^i_s \tag{II.9b}$$

$$p^- = \frac{1}{2\alpha' p^+} [\alpha'(p^i)^2 + 2\alpha'(\tilde{p}^I)^2 + 2\alpha'(\bar{p}^I)^2 + 2N + 2\tilde{N} - 1] \tag{II.10}$$

$$N = \sum_{n=1}^{\infty} (\alpha^i_{-n}\alpha^i_n + \frac{1}{2} n \bar{S}_{-n}\gamma^- S_n) \tag{II.11}$$

$$\tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \sum_{s=\frac{1}{2}}^{\infty} s b^i_{-s} b^i_s . \tag{II.12}$$

The constraint between left and right movers is

$$\frac{1}{2}\alpha' m_4^2 = N + \tilde{N} - \frac{1}{2} + \alpha'(\bar{p}^I)^2 + \alpha'(\tilde{p}^I)^2 . \tag{II.13}$$

Compactification with a Nonabelian Group

The left and right momenta $\bar{p}^I$ and $\tilde{p}^I$ reflect the degrees of freedom of the internal momentum and winding number.[46] From the requirement of torus compactification and in anticipation of the Neveu-Schwarz affine construction, we set the size of the internal dimensions R equal to $\sqrt{\alpha'}$, and we take the momenta in the compact dimensions to be quantized on the (self-dual) hypercubic lattice $Z^6$ with basis vectors of unit length squared:

$$\sqrt{2\alpha'}\tilde{p}^I = \sum_{L=1}^{d} N^L \alpha^I_L \tag{II.14a}$$

$$\sqrt{2\alpha'}p^I = \sum_{L=1}^{d} N^L \alpha_L^I \tag{II.14b}$$

$\alpha_L^I = \delta_L^I$, $N^L$ are integers.

The gauge algebra is given in terms of the operators $\tilde{\alpha}_o^I$ and $X_o(r)$ - this second operator will be defined below. These generators form a subalgebra of the affine $SU(2)^6$ constructed from Neveu-Schwarz operators. In order for the gauge generators to commute with the Lorentz generators in the uncompactified dimensions, the affine generators must be given in terms of a vertex operator of conformal spin one. (The total conformal spin of closed string vertex operators must be equal to two, but for the moment we are just going to look at the left-moving half). Therefore, in contrast with the Frenkel-Kac-Segal (FKS) construction in terms of the tachyon vertex operator of the VM, we define a new construction by using the tachyon operator of the NS model:

$$\begin{aligned} V(r,z) &= r^I \sum_{S=-\infty}^{\infty} b_S^I z^{-S} :e^{\frac{2ir^I}{\sqrt{2\alpha'}}\tilde{X}^I(z)}: \\ &= r^I \sum_{S=-\infty}^{\infty} b_S^I z^{-S}\, V_o(r,z) \end{aligned} \tag{II.15}$$

where $(r^I)^2 = 1$ and the $r^I$ are the roots of $SU(2)^6$. From eq. (II.2b) with $z = \exp[2i(\tau+\sigma)]$,

$$:e^{\frac{2ir^I}{\sqrt{2\alpha'}}\tilde{X}^I(z)}: = \exp(r^I \sum_{n=1}^{\infty} \frac{1}{n} \tilde{\alpha}^I_{-n} z^n)\, e^{\frac{2ir^I}{\sqrt{2\alpha'}}\tilde{x}^I}\, z^{\sqrt{2\alpha'}r^I\tilde{p}^I}$$

$$\cdot \exp(-r^I \sum_{n=1}^{\infty} \frac{1}{n} \tilde{\alpha}^I_n z^{-n})\, z^{\frac{(r^I)^2}{2}} \quad . \qquad \text{(II.16)}$$

The affine generators are $\tilde{\alpha}^I_n$ and $X_n(r)$ where

$$X_n(r) = \frac{c_r}{2\pi i} \oint \frac{dz}{z} z^n V(r,z). \qquad \text{(II.17)}$$

This is well-defined when the integrand is a single-valued function of complex z. Since $r^2 = 1$ and $s \in Z + \frac{1}{2}$, this requires that $r\cdot\sqrt{2\alpha'}\tilde{p}$ must be an integer, which is true for all the eigenvalues $\sqrt{2\alpha'}\tilde{p}^I \in Z^6$ (see eq. (II.14a)).

The affine Kac-Moody Lie algebra of $SU(2)^6$ with k = 1, but level x = 2 (the level of a Kac-Moody algebra is defined in eq. (IV.13)) is:

$$\begin{aligned}
&[X_n(r),X_m(r')] = 0 \qquad \text{for } r\cdot r' = 0,1 \\
&[X_n(r),X_m(-r)] = r\cdot\tilde{\alpha}_{n+m} + n\delta_{n,-m} \\
&[\tilde{\alpha}^I_n,\tilde{\alpha}^J_m] = n\delta_{n,-m}\delta^{IJ} \\
&[\tilde{\alpha}^I_n,X_m(r)] = r^I X_{n+m}(r).
\end{aligned} \qquad \text{(II.18)}$$

The root-dependent coefficients $c_r$ in eq. (II.17) are operator cocycles which satisfy

$$\begin{aligned}
&c_r c_{r'} = (-1)^{r\cdot r'+1} c_{r'} c_r \\
&c_r c_{-r} = -1
\end{aligned}$$

$$c_r|k\rangle = \epsilon(r,k)|r+k\rangle$$

$$\epsilon(r,r'+r'')\epsilon(r',r'') = \epsilon(r,r')\epsilon(r+r',r'') \qquad \text{(II.19)}$$

where $\epsilon(r,\tilde{k}) = \pm 1$. The first equation in (II.19) insures the hermiticity conditions $X_n^\dagger(r) = X_{-n}(-r)$ along with $\tilde{\alpha}_n^{I\dagger} = \tilde{\alpha}_{-n}^I$.

The SO(3,1) Lorentz generators are given as a subgroup of the ten-dimensional Lorentz algebra. They are $M^{+-}$, $M^{\hat{i}\hat{j}}$, and $M^{\hat{i}\pm}$ where

$$M^{+-} = \ell^{+-}$$

$$M^{\hat{i}+} = \ell^{\hat{i}+}$$

$$M^{\hat{i}\hat{j}} = \ell^{\hat{i}\hat{j}} - i\sum_{n=1}^{\infty} \frac{1}{n}(\tilde{\alpha}_{-n}^{\hat{i}}\tilde{\alpha}_n^{\hat{j}} - \tilde{\alpha}_{-n}^{\hat{j}}\tilde{\alpha}_n^{\hat{i}}) - i\sum_{s=1/2}^{\infty} (b_{-s}^{\hat{i}}b_s^{\hat{j}} - b_{-s}^{\hat{j}}b_s^{\hat{i}})$$
$$- i\sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^{\hat{i}}\alpha_n^{\hat{j}} - \alpha_{-n}^{\hat{j}}\alpha_n^{\hat{i}}) + \frac{i}{8}\sum_{n=-\infty}^{\infty} \bar{S}_{-n}\gamma^{\hat{i}\hat{j}-}S_n$$

$$M^{\hat{i}-} = \ell^{\hat{i}-} - i\sum_{n=1}^{\infty} \frac{1}{n}(\tilde{\alpha}_{-n}^{\hat{i}}\tilde{\alpha}_n^{-} - \tilde{\alpha}_{-n}^{-}\tilde{\alpha}_n^{\hat{i}}) - \frac{i}{\sqrt{2\alpha'}p^+}\sum_{s=1/2}^{\infty} (b_{-s}^{\hat{i}}G_s - G_{-s}b_s^{\hat{i}})$$
$$- i\sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^{\hat{i}}\alpha_n^{-} - \alpha_{-n}^{-}\alpha_n^{\hat{i}}) + \frac{i}{8\sqrt{2\alpha'}p^+}\sum_{n=-\infty}^{\infty}\ \sum_{m=-\infty}^{\infty} \bar{S}_{-n}\gamma^{\hat{i}\hat{j}-}S_n$$

$$\ell^{\mu\nu} = x^\mu p^\nu - p^\nu x^\mu$$

$$G_s = \sum_{m=-\infty}^{\infty} b_{s-m}^{i}\tilde{\alpha}_m^{i}. \qquad \text{(II.20)}$$

The SO(3,1) generators commute with the $SU(2)^6$ gauge generators $\tilde{\alpha}_0^I$ and $X_0(r)$ since the NS vertex operator of eq. (II.15) has conformal spin 1.

$$[\hat{L}_n, V(r,z)] = z\frac{d}{dz}(z^n V(r,z)) \tag{II.21a}$$

$$\begin{aligned}\{G_s, V(r,z)\} &= [\hat{L}_{2s}, V_0(r,z)] \\ &= z\frac{d}{dz}(z^{2s} V_0(r,z))\end{aligned} \tag{II.21b}$$

so that

$$[M^{\hat{i}-}, \oint \frac{dz}{z} V(r,z)] = 0. \tag{II.22}$$

The ten-dimensional supersymmetry charge is constructed from the right-moving superstring operators as[7]

$$Q^a = i(2\alpha')^{\frac{1}{4}}(p^+)^{\frac{1}{2}}(\gamma_+ S_0)^a + (2\alpha')^{\frac{-1}{4}}(p^+)^{\frac{-1}{2}} \sum_{n=-\infty}^{\infty} (\gamma_i S_{-n})^a \alpha_n^i \tag{II.23}$$

and reduces to four Majorana-spinor supersymmetry generators in four dimensions.

The physical spectrum consists of direct product states $(|\ \rangle_R \times |\ \rangle_L)$ of the Fock space of the right-moving superstring and the left-moving bosonic NS string subject to eq. (II.12). The constraint eliminates the tachyon. At the massless level there are four spin $\pm\frac{1}{2}$ fermions $|a\rangle$, so

$$|\ \rangle_R = |\hat{i}\rangle,\ |I\rangle,\ |a\rangle$$

and

$$|\ \rangle_L = b^{\hat{j}}_{-\frac{1}{2}}|0\rangle,\ b^{J}_{-\frac{1}{2}}|0\rangle,\ |\underline{k}^I\rangle$$

where $\sqrt{2\alpha'}\tilde{k}^I$ are the twelve roots of $SU(2)^6$ (with $\alpha'(\tilde{p}^I)^2 = \frac{1}{2}$). The physical massless states are

$$\{|\hat{\imath}\rangle_R, \ |I\rangle_R, \ |a\rangle_R\} \times b^{\hat{\jmath}}_{-\frac{1}{2}}|0\rangle$$

which is the d=4, N=4 supergravity multiplet with spin content $(\pm 2, 4(\pm\frac{3}{2}), 6(\pm 1), 4(\pm\frac{1}{2}), 2(0))$ and

$$\{|\hat{\imath}\rangle_R, \ |I\rangle_R, \ |a\rangle_R\} \times \{b^J_{-\frac{1}{2}}|0\rangle, \ |\tilde{k}^I\rangle\}$$

which is the d=4, N=4 supersymmetric Yang-Mills multiplet with spin content $(\pm 1, 4(\pm\frac{1}{2}), 6(0))$ in the adjoint of $SU(2)^6$.

Tree and One-Loop Amplitudes

To discuss the interacting string, we define the vertex operators for the emission of the massless closed string states in the four-dimensional model. This is done in analogy with the Virasoro-Shapiro (VS) tachyon vertex operator

$$V(k,z) = \int_0^\pi d\sigma \ W(k,z,\sigma)$$

which can be understood as the vertex for the emission of a ground state (zero number operator) with momentum $\bar{k}^i$ and $\tilde{k}^i$ each equal to $\frac{k^i}{2}$.

$$W(k,z,\sigma) = g \ :e^{ik^iX^i(\sigma,\tau)}: \ = g \ :e^{2i\bar{k}^i\bar{X}^i}: \ :e^{2i\tilde{k}^i\tilde{X}^i}:$$

where
$$X^i(\sigma,\tau) = \bar{X}^i + \tilde{X}^i$$
$$\bar{X}^i(\tau-\sigma) = \frac{1}{2}x^i + \alpha' p^i(\tau-\sigma) + \frac{i\sqrt{2\alpha'}}{2} \sum_{n\neq 0} \frac{1}{n} \alpha_n^i e^{-2in(\tau-\sigma)}$$
etc.

We can now write the vertex operator for states with nonzero winding number in a compactified closed string; the only difference between the VS vertex and this one is that the momenta $\bar{k}^I$ and $\tilde{k}^I$ for the internal coordinates no longer need to be equal.[10,46] Also, the left-moving part is written in the NSR formalism while the right-moving part is in terms of superstring operators.

The vertex operator for the massless boson states $|i\rangle_R \times b^j_{-\frac{1}{2}}|0\rangle_L$ is given by

$$V(k^{\hat{i}},z) = \frac{1}{\pi}\,\epsilon^{ij}\int_0^\pi d\sigma\, W^{ij}(k^{\hat{i}},z,\sigma)$$
$$W^{ij}(k^{\hat{i}},z,\sigma) = g\Big(\frac{1}{\sqrt{2\alpha'}} \sum_n \alpha_n^i \exp[-2in(\tau-\sigma)] + \frac{1}{2}k^{\hat{k}}R^{i\hat{k}}\Big)$$
$$\cdot\Big(\frac{1}{\sqrt{2\alpha'}} \sum_n \tilde{\alpha}_n^j \exp[-2in(\tau+\sigma)] - \frac{1}{2}k^{\ell} \sum_{s,r} b^j_s b^{\ell}_r \exp[-2i(s+r)(\tau+\sigma)]\Big)$$
$$\cdot\exp[ik^{\hat{i}}X^{\hat{i}}(\sigma,\tau)]. \qquad \text{(II.24)}$$

For all states $|i\rangle_R \times b^{\hat{j}}_{-\frac{1}{2}}|0\rangle_L$, $|\hat{i}\rangle_R \times b^J_{-\frac{1}{2}}|0\rangle_L$, etc., $\tilde{k}^I = \bar{k}^I = 0$.

Here
$$R^{ik} = \frac{1}{8}\bar{S}^a(\tau-\sigma)(\gamma^{ik-})^{ab}S^b(\tau-\sigma)$$

and from eqs. (II.1a) and (II.2a) $X^{\hat{i}}(\sigma,\tau) = X^{\hat{i}}(\tau-\sigma) + \tilde{X}^{\hat{i}}(\tau+\sigma)$. The polarizations and momenta are transverse:

$$k^{\hat{i}}\epsilon^{\hat{i}j} = \epsilon^{i\hat{j}}k^{\hat{j}} = 0$$

$$\epsilon^{\pm i} = k^{\pm} = 0 \qquad\qquad \text{etc.}$$

The vertex operators for the massless $SU(2)^6$ "charged" vector bosons $|\hat{i}\rangle\times|\tilde{k}^J\rangle$ and scalars $|I\rangle\times|\tilde{k}^J\rangle$ are

$$V(\tilde{k}^j,z) = \frac{1}{\pi}\epsilon^i \int_0^{\pi} d\sigma\, W^i(\tilde{k}^j,z.\sigma)$$

where

$$W^i(\tilde{k}^j,z,\sigma) = g\left(\frac{1}{\sqrt{2\alpha'}} \sum_n \alpha_n^i \exp[-2in(\tau-\sigma)] + \frac{1}{2}k^{\tilde{k}}R^{i\tilde{k}}\right) \exp[ik^{\hat{i}}X^{\hat{i}}(\sigma,\tau)]$$
$$\cdot\left(\tilde{k}^j \sum_s b_s^j \exp[-2is(\tau+\sigma)]\right) :\exp[2i\tilde{k}^I\tilde{X}^I(\tau+\sigma)]: c_{\tilde{k}} \qquad \text{(II.25)}$$

$$\tilde{k}^j = (\tfrac{1}{2}k^{\hat{j}},\tilde{k}^J)$$

and $:\exp(2i\tilde{k}^I\tilde{X}^I):$ is given in eq. (II.16) with $r^I = \sqrt{2\alpha'}\tilde{k}^I$. The vertices of eqs. (II.24) and (II.25) corresponding to the gauge bosons transform in the adjoint representation. The fermion vertex operators can be obtained from these by supersymmetry. The propagator is derived from the mass formula, eq. (II.13)

$$\Delta_c = \frac{\alpha'}{2\pi}\int_D d^2z\; z^{N + \alpha'(\bar{p}^I)^2}\bar{z}^{\tilde{N} - \frac{1}{2} + \alpha'(\tilde{p}^I)^2}|z|^{\frac{\alpha' p^2}{2} - 2} \; . \qquad \text{(II.26)}$$

The integration region D will be adjusted in the loop amplitudes to cover the torus once.

The only nonvanishing three-point gauge coupling is for the amplitude involving one neutral state $|\hat{\imath}\rangle_R \times b^I_{-\frac{1}{2}}|0\rangle_L$ and two charged states $|\hat{\imath}\rangle_R \times |\hat{k}^J\rangle_L$. In tree approximation, from eq. (II.24)

$$A^I(k_1^{\hat{k}},\hat{k}_1^K;k_2^{\hat{\imath}};k_3^{\hat{\imath}},\hat{k}_3^L) = \langle\epsilon_1;k_1^{\hat{k}},-\hat{k}_1^K|\ \frac{1}{\pi}\epsilon_2^{\hat{\imath}} \int_0^\pi d\sigma\ W^{\hat{\imath}I}(k_2,1,\sigma)\ |\epsilon_3;k_3^{\hat{\imath}};\hat{k}_3^L\rangle$$

$$= g\epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\zeta\ f^{I\,\hat{k}_1\,\hat{k}_3}\ [(k_2-k_1)^\mu\delta^{\nu\zeta} + (k_3-k_2)^\nu\delta^{\mu\zeta} + (k_1-k_3)^\zeta\delta^{\mu\nu}] \qquad \text{(II.27)}$$

with $f^{I\,\hat{k}_1\,\hat{k}_3} = \hat{k}_3^I\ \delta_{\hat{k}_1,-\hat{k}_3}$. Eq. (II.27) is the standard Yang-Mills three-point coupling for the gauge group $SU(2)^6$, and it proves that the affine algebraic compactification gives rise to standard interacting nonabelian gauge bosons.

The tree amplitude for four external charged vectors from eq. (II.25) is

$$A_{tree}(1234) = \langle\epsilon_1;-k_1|\ \epsilon_2^{\hat{\imath}}W^{\hat{\imath}}(k_2,1,0)\Delta_c\epsilon_3^{\hat{\jmath}}W^{\hat{\jmath}}(k_3,1,0)|\epsilon_4;k_4\rangle$$

$$= g^2\frac{1}{32}\alpha'\epsilon K\ \frac{\Gamma(\frac{-1}{4}\alpha's - \alpha'S)\ \Gamma(\frac{-1}{4}\alpha't - \alpha'T)\ \Gamma(\frac{-1}{4}\alpha'u - \alpha'U)}{\Gamma(\frac{1}{4}\alpha's + 1)\ \Gamma(\frac{1}{4}\alpha't + 1)\ \Gamma(\frac{1}{4}\alpha'u + 1)} \qquad \text{(II.28)}$$

where $$\epsilon = (-1)^{2\alpha'\tilde{k}_1\cdot\tilde{k}_3}\epsilon(2,3+4)\epsilon(3,4) \tag{II.29}$$

is formed from the operator cocycles evaluated on the states, and K is the superstring kinematic factor given by Schwarz in eq. (4.24) of ref. [7].

The Mandelstam variables are

$$\begin{aligned} s &= -(k_1^{\hat{i}} + k_2^{\hat{i}})^2 \quad & t &= -(k_2^{\hat{i}} + k_3^{\hat{i}})^2 \quad & u &= -(k_1^{\hat{i}} + k_3^{\hat{i}})^2 \\ S &= -(\tilde{k}_1^I + \tilde{k}_2^I)^2 \quad & T &= -(\tilde{k}_2^I + \tilde{k}_3^I)^2 \quad & U &= -(\tilde{k}_1^I + \tilde{k}_3^I)^2 \end{aligned} \tag{II.30}$$

and $$s + t + u = 0$$
$$\alpha'(S + T + U) = -2.$$

The allowed values of $\alpha'S$, etc. are 0,-1,-2, so eq. (II.28), which is totally symmetric under the interchange of any two external particles, has no tachyon pole.

Finally, we briefly outline our calculation of the one-loop amplitudes. For external charged vector mesons, using eq. (II.25), we get:

$$\begin{aligned} A_{loop}(1234) &= \epsilon_1^{\hat{i}}\epsilon_2^{\hat{j}}\epsilon_3^{\hat{k}}\epsilon_4^{\hat{l}} \int dp \; \mathrm{tr}\, (\Delta_c W^{\hat{i}}(1)\Delta_c W^{\hat{j}}(2)\Delta_c W^{\hat{k}}(3)\Delta_c W^{\hat{l}}(4)) \\ &= (\frac{\alpha' g}{2\pi})^4 \frac{1}{16\alpha'^2} K\, \tilde{\epsilon}\, (2\pi)^8 \int d^2\tau \int \prod_{M=1}^{3} d^2\nu_M \,(\mathrm{Im}\,\tau)^{-2} \prod_{1\le I<J\le 4} (\chi_{IJ})^{\alpha' k_I^{\hat{k}} k_J^{\hat{k}}} \\ &\cdot \exp(i\pi\bar{\tau})\, f[e^{-2\pi i\bar{\tau}}]^{-8} \prod_{I<J} (\Psi_{IJ})^{2\alpha'\tilde{k}_I^K\tilde{k}_J^K}\, \Omega\, \Omega'\, \phi(\exp(-2\pi i\bar{\tau}))^8 \\ &\cdot [(\tilde{k}_1^i\tilde{k}_2^i)(\tilde{k}_3^j\tilde{k}_4^j)X_{43}^+X_{21}^+ - (\tilde{k}_1^i\tilde{k}_3^i)(\tilde{k}_2^j\tilde{k}_4^j)X_{42}^+X_{31}^+ + (\tilde{k}_1^i\tilde{k}_4^i)(\tilde{k}_2^j\tilde{k}_3^j)X_{41}^+X_{32}^+] \end{aligned} \tag{II.31}$$

where $\tilde{\epsilon} = \epsilon(1,2+3+4+p)\epsilon(2,3+4+p)\epsilon(3,4+p)\epsilon(4,p)$

$$= \epsilon(2,3+4)\epsilon(3,4)$$

$$f(\bar{w} = \exp(-2\pi i\bar{\tau})) = \prod_{n=1}^{\infty} (1-\bar{w}^n)$$

$$\Psi_{IJ} = 2\pi i \exp[\frac{-i\pi}{\bar{\tau}}(\bar{\nu}_{JI})^2] \frac{\tilde{\theta}_1(\bar{\nu}|\bar{\tau})}{\tilde{\theta}_1'(0|\bar{\tau})}$$

$$\chi_{IJ} = 2\pi \exp(\frac{-\pi(\mathrm{Im}\ \nu_{JI})^2}{\mathrm{Im}\ \tau}) \left|\frac{\theta_1(\nu_{JI}|\tau)}{\theta_1'(0|\tau)}\right|$$

$$\nu_{JI} = \nu_J - \nu_I$$

$$\tilde{\theta}_1(\bar{\nu}|\bar{\tau}) = 2\bar{w}^{\frac{1}{8}} f(\bar{w}) \sin\pi\bar{\nu} \prod_{n=1}^{\infty} (1 - 2\bar{w}^n \cos 2\pi\bar{\nu} + \bar{w}^{2n})$$

$$\Omega = \sum_{\sqrt{2\alpha'}\tilde{p}^I \in Z^6} \exp[\alpha' \ln\bar{w} (\tilde{p}^I - \sum_{M=1}^{4} \frac{\ln\bar{z}}{\ln\bar{w}} Q_M^I)^2]$$

$$\Omega' = \sum_{\sqrt{2\alpha'}\tilde{p}^I \in Z^6} w^{\alpha'(\bar{p}^I)^2} = [\theta_3(0|\tau)]^6$$

$$Q_{M+1}^I = \sum_{N=1}^{M} k_N^I$$

$$\phi(\bar{w}) = \prod_{s=1/2}^{\infty} (1 + \bar{w}^s)$$

$$X_{JI}^+ = \sum_{s=1/2}^{\infty} \frac{\bar{c}_{JI}^s + (\bar{w}/\bar{c}_{JI})^s}{1 + \bar{w}^s}$$

$$\bar{c}_{JI} = e^{-2\pi i\bar{\nu}_{JI}}. \tag{II.32}$$

The integrand of eq. (II.31) is invariant under $\nu_I \to \nu_I + 1$, $\nu_I \to \nu_I + \tau$ and under the subgroup of modular transformations generated by $\tau \to \tau + 2$ and $\tau \to \frac{-1}{\tau}$. The other four-point external boson one-loop amplitudes can be calculated and can be shown to have the same theta subgroup of modular symmetry. The fundamental region for this theta subgroup is taken as:

$$\begin{aligned} F_{SM}: \qquad -1 < \mathrm{Re}\,\tau &< 1 \\ \mathrm{Im}\,\tau &> 0 \\ |\tau| &> 1. \end{aligned} \tag{II.33}$$

A Massive Ramond Sector

Now we want to look at the R×SS sector and merge this with the NS×SS model to form a fully modular invariant, four-dimensional NSR×SS model, which is a compactified version of the type II superstring.

In the Ramond sector, the two-dimensional, left-moving worldsheet fermions are integrally moded

$$d^i(\tau+\sigma) = \sum_{n=-\infty}^{\infty} d_n^i \, e^{-2in(\tau+\sigma)} \tag{II.34}$$

and the oscillators obey the anti-commutation relations

$$\{d_n^i, d_m^j\} = \delta_{n,-m}\delta^{ij}. \tag{II.35}$$

The zero modes thus form a Clifford algebra, and this corresponds to the fact that the Ramond ground state is a spacetime spinor.

The R×SS sector which is needed to give us a fully modular-invariant theory is one in which the internal momenta live on a shifted lattice - that is the R×SS fermionic string is compactified on a $(Z+\frac{1}{2})^6$ lattice. The allowed internal momenta for this sector cannot be zero and are given by

$$\sqrt{2\alpha'}\bar{p}^I = \sum_L (N + \tfrac{1}{2})^L \alpha_L^I \tag{II.36a}$$

$$\sqrt{2\alpha'}\tilde{p}^I = \sum_L (N + \tfrac{1}{2})^L \alpha_L^I . \tag{II.36b}$$

These shifted momenta correspond to a twisted boundary condition on the internal coordinates, i.e.

$$X^I(\sigma=0) = X^I(\sigma=\pi) + \sqrt{2\alpha'}\sum_L N^L \alpha_L^I \pi + \sqrt{2\alpha'}\,\tfrac{1}{2}\pi . \tag{II.37}$$

The lowest mass state in this sector is

$$e^{i\hat{x}^I \bar{p}_1^I}\{|\hat{\imath}\rangle_R,\ |I\rangle_R,\ |a\rangle_R\} \times e^{i\hat{x}^I \tilde{p}_1^I}\{|a\rangle_L\} \tag{II.38}$$

with $\qquad \alpha' m_4^2 = 3$

since the lowest momentum is $\sqrt{2\alpha'}p_1^I = (\frac{1}{2},\ldots,\frac{1}{2})$. The left-right constraint for this sector is

$$\bar{N} + \alpha'(\bar{p}^I)^2 = \tilde{\bar{N}} + \alpha'(\tilde{p}^I)^2 \tag{II.39}$$

and the mass operator is

$$\frac{1}{2}\alpha' m_4^2 = \bar{N} + \tilde{\bar{N}} + \alpha'(\bar{p}^I)^2\ \alpha'(\tilde{p}^I)^2 \tag{II.40}$$

where

$$\bar{N} = \sum_{n=1}^{\infty} (\alpha_{-n}^i \alpha_n^i + \frac{1}{2} n \bar{S}_{-n} \gamma^- S_n) \tag{II.41a}$$

$$\tilde{\bar{N}} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \sum_{n=1}^{\infty} n d_{-n}^i d_n^i . \tag{II.41b}$$

The propagator for the R×SS string is given by

$$\bar{\Delta}_c = \frac{\alpha'}{2\pi} \int_D d^2z\ z^{\bar{N} + \alpha'(\bar{p}^I)^2} \bar{z}^{\tilde{\bar{N}} + \alpha'(\tilde{p}^I)^2} |z|^{\frac{\alpha' p^2}{2} - 2} . \tag{II.42}$$

and the vertex operator for the emission of charged vector bosons $|\hat{i}\rangle_R \times |k^J\rangle_L$ from a fermion line in the compactified (massive) R×SS sector is given by:[45]

$$\bar{W}^{\hat{i}}(k^j, z, \sigma) = g\Big(\frac{1}{\sqrt{2\alpha'}} \sum_n \alpha_n^i \exp[-2in(\tau-\sigma)] + \frac{1}{2} k^{\hat{k}} R^{i\hat{k}}\Big) \exp[ik^{\hat{i}} X^{\hat{i}}(\sigma,\tau)]$$

$$\cdot \hat{k}^j [\gamma^j + i\sqrt{2}\gamma_9 \sum_{n\neq 0} d_n^j\, e^{-2in(\tau+\sigma)}]\ \gamma_9 (-1)^{\sum_{n=1}^{\infty} d_{-n}^i d_n^i}$$

$$\cdot \frac{1}{i\sqrt{2}} :e^{2ik^I X^I(\tau+\sigma)}:\ c_k . \tag{II.43}$$

The Ramond sector projection operator is

$$(-1)^{\bar{P}} = 2^4 d_o^1 d_o^2 \cdots d_o^8 \, (-1)^{\sum\limits_{n=1}^{\infty} d_{-n}^i d_n^i + 2\alpha'(\bar{p}^I)^2 + 2\alpha'(\tilde{p}^I)^2} \quad . \tag{II.44}$$

The projection $(-1)^{\bar{P}} = 1$ indicates that the left-moving Ramond fermions are Majorana-Weyl. This projection will not actually contribute to the 4-point, one-loop amplitude for the scattering of charged vector bosons, but it is required for the full unitarity of the one-loop amplitudes in the R×SS sector.

Rather than inserting this sector into the theory by hand, calculating one-loop amplitudes, and showing that the combined (NS+R)×SS theory is fully modular invariant, I will instead show that the full theory is contained in the NS sector alone. That is, I will show that the modular subgroup invariant amplitude of the NS×SS sector is equal to the amplitude of the complete (NS+R)×SS sector with the integration over the fundamental region of the full modular group and with the Ramond sector internal momenta defined on the shifted lattice. Let me rewrite the one-loop amplitude for the NS×SS sector alone:

$$A_{loop}(1234) = \epsilon_1^{\hat{i}}\epsilon_2^{\hat{j}}\epsilon_3^{\hat{k}}\epsilon_4^{\hat{l}} \int dp \; tr\,(\Delta_c W^{\hat{i}}(1)\Delta_c W^{\hat{j}}(2)\Delta_c W^{\hat{k}}(3)\Delta_c W^{\hat{l}}(4))$$

$$= \left(\frac{\alpha' g}{2\pi}\right)^4 \frac{1}{16\alpha'^2} K \tilde{\epsilon} \, (2\pi)^8 \int_{F_{SM}} d^2\tau (\mathrm{Im}\,\tau)^{-2} g(\tau) \tag{II.45}$$

where $F_{SM}$ is the fundamental region of the theta subgroup of the modular group defined in eq. (II.33), the $W^{\hat{i}}$'s are defined in eq. (II.33), and

$$g(\tau) = \int \prod_{M=1}^{3} d^2\nu_M \prod_{1\le I<J\le 4} (\chi_{IJ})^{\alpha' k_I^{\hat{k}} k_J^{\hat{k}}} (\bar{w})^{-\frac{1}{2}}$$
$$\cdot f(\bar{w})^{-8} \prod_{I<J} (\Psi_{IJ})^{2\alpha' \bar{k}_I^K \bar{k}_J^K} \,\Omega\, \Omega' \,\phi^+(\bar{w})^8$$
$$\cdot [(\bar{k}_1^i \bar{k}_2^i)(\bar{k}_3^j \bar{k}_4^j) X_{43}^+ X_{21}^+ - (\bar{k}_1^i \bar{k}_3^i)(\bar{k}_2^j \bar{k}_4^j) X_{42}^+ X_{31}^+ + (\bar{k}_1^i \bar{k}_4^i)(\bar{k}_2^j \bar{k}_3^j) X_{41}^+ X_{32}^+] \;. \quad \text{(II.46)}$$

The statement then is that the integration over the modular subgroup region can be rewritten as an integration over the full modular group fundamental region F:

$$\int_{F_{SM}} d^2\tau \,(\mathrm{Im}\,\tau)^{-2} g(\tau) = \int_F d^2\tau \,(\mathrm{Im}\,\tau)^{-2} [g(\tau) + g(\tau+1) + g(-\tfrac{1}{\tau}+1)] \;. \quad \text{(II.47)}$$

Using the transformation properties of the Jacobi theta functions,[47] we can calculate the two new terms $g(\tau+1)$ and $g(-\frac{1}{\tau}+1)$.

$$g(\tau+1) = \int \prod_{M=1}^{3} d^2\nu_M \prod_{1\le I<J\le 4} (\chi_{IJ})^{\alpha' k_I^{\hat{k}} k_J^{\hat{k}}} f(\bar{w})^{-8}$$
$$\cdot [-(\bar{w})^{-\frac{1}{2}} \prod_{s=1/2}^{\infty} (1-\bar{w}^s)^8 \sum_{\sqrt{2\alpha'}\bar{p}^I \,\epsilon\, Z^6} w^{\alpha'(\bar{p}^I)^2} (-1)^{2\alpha'(\bar{p}^I)^2}]$$
$$\cdot \prod_{I<J} (\Psi_{IJ})^{2\alpha' \bar{k}_I^K \bar{k}_J^K} \sum_{\sqrt{2\alpha'}\bar{p}^I \,\epsilon\, Z^6} \exp(\alpha' \ln\bar{w}\, [\tilde{p}^I - \sum_{M=1}^{4} \frac{\ln \bar{z}_M}{\ln \bar{w}} Q_M^I]^2 \,(-1)^{2\alpha'(\tilde{p}^I)^2})$$
$$\cdot [(\bar{k}_1^i \bar{k}_2^i)(\bar{k}_3^j \bar{k}_4^j) X_{43}^+ X_{21}^+ - (\bar{k}_1^i \bar{k}_3^i)(\bar{k}_2^j \bar{k}_4^j) X_{42}^+ X_{31}^+ + (\bar{k}_1^i \bar{k}_4^i)(\bar{k}_2^j \bar{k}_3^j) X_{41}^+ X_{32}^+] \quad \text{(II.48)}$$

$$g\left(-\frac{1}{\tau}+1\right) = \int \prod_{M=1}^{3} d^2\nu_M \prod_{1\leq I<J\leq 4} (\chi_{IJ})^{\alpha' k_I^{\hat{k}} k_J^{\hat{k}}} f(\bar{w})^{-8}$$

$$\cdot\left[-\frac{16}{\sqrt{2\alpha'}} \prod_{n=1}^{\infty} (1+\bar{w}^{n})^8 \sum_{\sqrt{2\alpha'}\bar{p}^I \,\epsilon\, (Z+\frac{1}{2})^6} w^{\alpha'(\bar{p}^I)^2}\right]$$

$$\cdot \prod_{I<J} (\Psi_{IJ})^{2\alpha' \tilde{k}_I^K \tilde{k}_J^K} \sum_{\sqrt{2\alpha'}\bar{p}^I \,\epsilon\, (Z+\frac{1}{2})^6} \exp\left(\alpha' \ln\bar{w} \left[\tilde{p}^I - \sum_{M=1}^{4} \frac{\ln \bar{z}_M}{\ln \bar{w}} Q_M^I\right]^2 \right)$$

$$\cdot\left[(\tilde{k}_1^i\tilde{k}_2^i)(\tilde{k}_3^j\tilde{k}_4^j)\tilde{\chi}^+_{43}\tilde{\tilde{\chi}}^+_{21} - (\tilde{k}_1^i\tilde{k}_3^i)(\tilde{k}_2^j\tilde{k}_4^j)\tilde{\chi}^+_{42}\tilde{\tilde{\chi}}^+_{31} + (\tilde{k}_1^i\tilde{k}_4^i)(\tilde{k}_2^j\tilde{k}_3^j)\tilde{\chi}^+_{41}\tilde{\tilde{\chi}}^+_{32}\right] . \quad \text{(II.49)}$$

The functions $\chi^+_{JI}$, $\tilde{\chi}^+_{JI}$, and $\tilde{\tilde{\chi}}^+_{JI}$ are

$$\chi^+_{JI} = \sum_{s=1/2}^{\infty} \frac{\bar{c}^s_{JI} + (\bar{w}/\bar{c}_{JI})^s}{1 + \bar{w}^s}$$

$$= \frac{1}{2} i\ \bar{\theta}_2(0|\tau)\ \bar{\theta}_4(0|\tau)\ \frac{\bar{\theta}_3(\nu_{JI}|\tau)}{\bar{\theta}_1(\nu_{JI}|\tau)} \quad \text{(II.50a)}$$

$$\tilde{\chi}^+_{JI} = \sum_{s=1/2}^{\infty} \frac{\bar{c}^s_{JI} - (\bar{w}/\bar{c}_{JI})^s}{1 - \bar{w}^s}$$

$$= \frac{1}{2} i\ \bar{\theta}_2(0|\tau)\ \bar{\theta}_4(0|\tau)\ \frac{\bar{\theta}_4(\nu_{JI}|\tau)}{\bar{\theta}_1(\nu_{JI}|\tau)} \quad \text{(II.50b)}$$

$$\tilde{X}^{+}_{JI} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{\bar{c}^{n}_{JI} + (\bar{w}/\bar{c}_{JI})^{n}}{1 + \bar{w}^{n}}$$

$$= \frac{1}{2} i \; \bar{\theta}_{3}(0|\tau) \; \bar{\theta}_{4}(0|\tau) \; \frac{\bar{\theta}_{2}(\nu_{JI}|\tau)}{\bar{\theta}_{1}(\nu_{JI}|\tau)} \; . \qquad \text{(II.50c)}$$

We can now identify the contribution coming from the first two terms on the right of eq. (II.47) as that of a projected NS×SS string:

$$\int_{F} d^{2}\tau \; (\mathrm{Im}\;\tau)^{-2} \; \left(\frac{g(\tau) + g(\tau+1)}{2}\right) = (\text{constant}) \int dp \; \mathrm{tr} \; \left(\frac{1 + (-1)^{P}}{2}\right)\Delta_{c}W^{\hat{i}}(1)$$

$$\left(\frac{1 + (-1)^{P}}{2}\right)\Delta_{c}W^{\hat{j}}(2)\left(\frac{1 + (-1)^{P}}{2}\right)\Delta_{c}W^{\hat{k}}(3)\left(\frac{1 + (-1)^{P}}{2}\right)\Delta_{c}W^{\hat{l}}(4) \qquad \text{(II.51)}$$

where $$P = \sum_{s=1/2}^{\infty} b^{i}_{-s} b^{i}_{s} - 1 + 2\alpha'(\bar{p}^{I})^{2} + 2\alpha'(\tilde{p}^{I})^{2}. \qquad \text{(II.52)}$$

Therefore $g(\tau) + g(\tau+1)$ is a projection onto states with $(-1)^{P} = 1$. This projection in fact does not eliminate any states from the original NS×SS spectrum since $(-1)^{P} = 1$ is automatically satisfied by the left-right constraint in eq. (II.12).

Similarly, the last term in eq. (II. 47) can be identified as coming from a massive Ramond sector. The shifted momentum lattice can be seen explictly in eq. (II.49). For $g(\frac{-1}{\tau}+1)$ we thus get:

$$\int_F d^2\tau\ (\mathrm{Im}\ \tau)^{-2}\ \left(\frac{g(-\frac{1}{\tau}+1)}{2}\right) = -(\mathrm{constant}) \int dp\ \mathrm{tr}\ \left(\frac{1+(-1)^{\bar{P}}}{2}\right)\bar{\Delta}_c\bar{W}^{\hat{\imath}}(1)$$

$$\cdot\left(\frac{1+(-1)^{\bar{P}}}{2}\right)\bar{\Delta}_c\bar{W}^{\hat{\jmath}}(2)\left(\frac{1+(-1)^{\bar{P}}}{2}\right)\bar{\Delta}_c\bar{W}^{\hat{k}}(3)\left(\frac{1+(-1)^{\bar{P}}}{2}\right)\bar{\Delta}_c\bar{W}^{\hat{l}}(4) \qquad \text{(II.53)}$$

where $\bar{\Delta}_c$, $\bar{W}^{\hat{\imath}}$, and $(-1)^{\bar{P}}$ were defined in eqs. (II.42-44).

The original amplitude thus corresponds to a fully modular-invariant string theory. In ten dimensions it is the type II superstring. In four dimensions it carries the gauge group $SU(2)^6$. The left-moving Ramond sector becomes massive, but the full theory is still N=4 supersymmetric in four dimensions because the right-moving supersymmetry remains unbroken. This theory was the first example of a compactified type II superstring theory that is fully modular invariant and that incorporates a nonabelian gauge group. In section IV this theory will be generalized so as to include rank 4 realistic gauge groups large enough to contain the gauge group of the standard model.

Unitarity

We can now verify the unitarity of the theory. We do this by calculating the imaginary part of eq. (II.45) and showing that it is equal to a product of tree amplitudes. Since eq. (II.45) can be expressed as the sum of eqs. (II.51) and (II.53), the discontinuity can be calculated in the standard way using the Cutkosky prescription[48-49] of replacing propagators with delta functions, and we find that the imaginary part of eq. (II.45) is an infinite sum of theta step functions, multiplied by tree amplitudes, with thresholds

given by the free particle spectrum of the compactified NS×SS and R×SS sectors.

The fact that the amplitude has been rewritten in a form that is modular invariant simplifies the calculation because the contribution to the imaginary part of the amplitude comes from the region $\mathrm{Im}\ \tau \to \infty$,[50] while if we had tried to verify unitarity for the NS theory alone we would have had to consider the additional contributions coming from the cusps at $|\tau| = 1$, $\mathrm{Re}\ \tau = 0$ in the fundamental region of the theta subgroup. These additional contributions correspond in the transformed picture to the projected NS and R sectors.

## III. Superconformal Invariance

In the operator formalism, the reparametrization invariance of the Nambu action permits us to choose a particular gauge, called the orthonormal gauge, which leads to a simplified equation of motion for the string.[40] The orthonormal gauge is obtained by imposing the two constraints:

$$\partial_\sigma X \cdot \partial_\tau X = 0 \tag{III.1}$$

$$\partial_\sigma X \cdot \partial_\sigma X + \partial_\tau X \cdot \partial_\tau X = 0. \tag{III.2}$$

The Virasoro operators are formed from the Fourier components of the left-hand sides of these equations, and we find that classically the $L_n$ operators must vanish

$$L_n = \int_{-\pi}^{-\pi} e^{in\sigma}(\partial_\tau X + \partial_\sigma X)^2 d\sigma. \tag{III.3}$$

For the supersymmetric theory, we may similarly take the Fourier modes of the supersymmetry current to obtain the super-Virasoro $G_s$ operators in the case of NS boundary conditions or the $F_m$ operators in the case of R boundary conditions. For the NS case, the $L_n$ and $G_s$ operators form a closed algebra called the super-Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\hat{c}}{8}n(n^2-1)\delta_{n,-m} \tag{III.4a}$$

$$[L_n, G_s] = (\frac{n}{2}-s)G_{n+s} \tag{III.4b}$$

$$\{G_r,G_s\} = 2L_{r+s} + \frac{\hat{c}}{2}(r^2-\frac{1}{4})\delta_{r,-s}. \qquad \text{(III.4c)}$$

In light-cone gauge, these operators are defined in terms of the string oscillators

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} \alpha^i_{n-m}\alpha^i_m + \frac{1}{2}\sum_{s=-\infty}^{\infty}(s-\frac{n}{2})b^i_{n-s}b^i_s \qquad \text{(III.5)}$$

$$G_s = \sum_{n=-\infty}^{\infty} \alpha^i_n b^i_{s-n} \qquad \text{(III.6)}$$

$$i=1,2,\ldots,D-2$$

and the closure of the Lorentz algebra requires that the central charge $\hat{c}$ take tha value

$$\hat{c} = D-2 = 8 \qquad \text{(in light-cone gauge)}.$$

All of this follows from the Nambu action with the orthonormal constraints, eqs. (III.1) and (III.2). This is not the most general action, however. Following the work of Polyakov, we can write down an action that includes the worldsheet metric $h^{\alpha\beta}$ and a gravitino field $\chi_{A\alpha}$:[41,4]

$$S = \frac{-1}{2\pi}\int d^2\sigma\, e[h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu - i\bar{\psi}^\mu\zeta^\alpha\partial_\alpha\psi_\mu + 2\bar{\chi}_\alpha\zeta^\beta\zeta^\alpha\psi^\mu\partial_\beta X_\mu + \frac{1}{2}\bar{\psi}_\mu\psi^\mu\chi_\alpha\zeta^\beta\zeta^\alpha\chi_\beta] \qquad \text{(III.7)}$$

where

$$e = \sqrt{h}$$

$$h = -\det h_{\alpha\beta}$$

$$h_{\alpha\beta} = \eta_{mn}e^m_\alpha e^n_\beta$$

$e^m_\alpha$ = vielbein fields

$\zeta^\alpha$ = 2-dimensional Dirac matrices

$\psi^\mu$ = NS fermions.

This action is invariant under local supersymmetry and reparametrizations. So instead of arbitrarily postulating the super-Virasoro constraints, we can derive them directly from the action, and in this way the theory is elevated to the level of a gauge theory. The super-Virasoro constraints arise in this picture by gauge fixing a two-dimensional Lagrangian with local supersymmetry and general coordinate covariance.

The energy-momentum tensor and supercurrent are obtained from the action by

$$T_{\alpha\beta} = -2\pi\frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}} \tag{III.8a}$$

$$J_\alpha = \frac{-\pi}{2e}\frac{\delta S}{\delta \chi^\alpha} \tag{III.8b}$$

which are evaluated in the gauge $e^n_\alpha = \delta^n_\alpha$ and $\chi_\alpha = 0$.

The energy-momentum tensor derived in this way is traceless, and its conservation equation can be used to show that its components are purely analytic or anti-analytic functions. In this way we find that $T_{\alpha\beta}$ generates conformal transformations on the complex plane associated with the worldsheet, and that the string theory can be reformulated as a 2-dimensional conformal field theory. Finally, we obtain a supersymmetric extension, or superconformal theory, by including the supercurrent in the algebra.[51-52,18-19]

Conformal Field Theory

Conformal symmetry is the symmetry of conformal mappings of the two-dimensional plane.[4,18] The mapping of the worldsheet coordinates into the complex plane is accomplished by

$$z = \exp(\tau + i\sigma). \tag{III.9}$$

Lines of constant $\tau$ are mapped into circles on the z-plane, and time ordering is replaced by radial ordering. In terms of the z-coordinates the metric is given by

$$g_{z\bar{z}} = \frac{1}{2} \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad . \tag{III.10}$$

The tracelessness of the energy-momentum tensor becomes

$$T_{z\bar{z}} = 0 \tag{III.11}$$

and the conservation equation is

$$\partial_{\bar{z}} T_{zz} + \partial_z T_{\bar{z}z} = 0. \tag{III.12}$$

These then imply

$$\partial_{\bar{z}} T_{zz} = 0. \tag{III.13}$$

Thus, $T(z) = T_{zz}$ is an analytic function, while $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}$ is anti-analytic. It can be shown that $T(z)$ generates conformal transformations. Infinitesimally, such a transformation is given by

$$z \to z + \epsilon(z) \qquad \text{(III.14)}$$

and is generated by

$$T_\epsilon = \oint \frac{dz}{2\pi i}\, \epsilon(z)T(z) \qquad \text{(III.15)}$$

where the contour is around the origin. In a quantum theory, the transformation (III.14) on a field is implemented by taking the commutator of the field with $T_\epsilon$. For example, suppose we would like to know how a field $\phi(z)$ transforms. We would have to take the equal-time commutator of $\phi(z')$ with $T(z)$, which is defined as the difference of two correlation functions in which the operator $T_\epsilon$ has been displaced slightly forward and backward in time (that is, radially from the origin). The commutator of the Fourier components may be obtained by taking a second contour integral with respect to the $z'$ variable. The difference of the two double contours may then be deformed into a single integral of $z$ about $z'$ followed by the integral of $z'$ around the origin:

$$\oint \frac{dz'}{2\pi i}\,[T_\epsilon, \phi(z')] = [\underset{z>z'}{\oint_0 \frac{dz'}{2\pi i} \oint_0 \frac{dz}{2\pi i}} - \underset{z'>z}{\oint_0 \frac{dz'}{2\pi i} \oint_0 \frac{dz}{2\pi i}}]\epsilon(z)T(z)\phi(z')$$

$$= \oint_0 \frac{dz'}{2\pi i} \oint_{z'} \frac{dz}{2\pi i}\, \epsilon(z)T(z)\phi(z'). \qquad \text{(III.15)}$$

On the right-hand side, the operator product $T(z)\phi(z')$ is understood to be radially ordered. Because the contour for $z$ encircles $z'$, the only contributions are going to come from the singular terms in $T(z)\phi(z')$ as $z \to z'$.

Thus, the operator product expansion (OPE) of $T(z)\phi(z')$ will contain all of the same information as the commutator, and hence we need only study the OPE's from this point on.

Operator Product Expansions

If the operators $T(z)$ and $\phi(z')$ are given as normal-ordered products of fields, then finding the singular terms of the time-ordered product $T(z)\phi(z')$, which is defined as the operator product expansion (OPE) (or Wilson expansion), amounts to nothing more than keeping the singular terms of the Wick contraction of $T(z)\phi(z')$ as $z \to z'$.[18]

For example, consider the case where the energy-momentum tensor is given in terms of fields of conformal spin 1:

$$T(z) = \tfrac{1}{2}\alpha^i(z)\alpha^i(z) \qquad \text{(III.16)}$$
$$i = 1,2,\ldots,D,$$

and let us find the OPE of this with itself. The 2-point correlation function for the $\alpha^i(z)$ fields is

$$\langle\alpha^i(z)\alpha^j(z')\rangle = \frac{\delta^{ij}}{(z-z')^2}. \qquad \text{(III.17)}$$

The OPE then consists of the singular terms in the Wick contraction of $T(z)T(z')$:

$$T(z)T(z') = \frac{1}{4}\, \alpha^i(z)\alpha^i(z)\alpha^j(z')\alpha^j(z')$$

$$= \frac{1}{2}\, \frac{\delta^{ij}}{(z-z')^2}\, \frac{\delta^{ij}}{(z-z')^2} + \frac{\delta^{ij}}{(z-z')^2}\, \alpha^i(z)\alpha^j(z'). \qquad \text{(III.18)}$$

Expand $\alpha^i(z)$ in a Taylor series

$$\alpha^i(z) = \alpha^i(z') + \partial_{z'}\alpha^i(z')(z-z') + \cdot\cdot\cdot \qquad \text{(III.19)}$$

and use $\delta^{ij}\delta^{ij} = D$, to get

$$T(z)T(z') = \frac{\frac{D}{2}}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial_{z'}T(z')}{(z-z')}. \qquad \text{(III.20)}$$

Defining the Fourier modes of $T(z)$ as

$$L_n = \frac{1}{2\pi i} \oint \frac{dz}{z}\, T(z)\, z^{n+2} \qquad \text{(III.21)}$$

we get

$$[L_n, L_m] = \frac{1}{2\pi i} \oint \frac{dz'}{z'}\, z'^{\,m+2}\, \frac{1}{2\pi i} \oint \frac{dz}{z}\, z^{n+2}\, T(z)T(z')$$

$$[L_n, L_m] = \frac{1}{2\pi i} \oint dz'\ z'^{m+1} \frac{1}{2\pi i} \oint dz\ z^{n+1} [\frac{\frac{D}{2}}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial_{z'} T(z')}{(z-z')}]$$

$$= \frac{1}{2\pi i} \oint dz'\ z'^{m+1} [\frac{(n+1)n(n-1)}{3\cdot 2} \frac{D}{2} z'^{n-2} + (n+1)2z'^{n} T(z')$$
$$+ z'^{n+1} \partial_{z'} T(z')]$$

$$= \frac{1}{2\pi i} \oint dz' [n(n^2-1)\frac{D}{12} z'^{m+n-1} + (2n+2-n-m-2) z'^{m+n+1} T(z')]$$

$$= \frac{D}{12} n(n^2-1)\delta_{n,-m} + (n-m)L_{n+m} \tag{III.22}$$

which is the Virasoro algebra of a bosonic string theory with c=D.

Super-Virasoro Algebras

The Virasoro algebra may be extended to a supersymmetric algebra by including the supercurrent G(z) (in the NS case). The corresponding string theory is then the spinning string, and T(z) and G(z) may be defined in terms of bosonic $\alpha^i(z)$ fields and anti-commuting fermionic fields $b^i(z)$:

$$T(z) = \tfrac{1}{2}\alpha^i(z)\alpha^i(z) + \tfrac{1}{2}\partial_z b^i(z) b^i(z)$$
$$G(z) = \alpha^i(z) b^i(z) \tag{III.23}$$

where in addition to eq. (III.17) we also have

$$\langle b^i(z)b^j(z')\rangle = \frac{\delta^{ij}}{(z-z')}$$
$$\langle \alpha^i(z)b^j(z')\rangle = 0. \tag{III.24}$$

Calculating the OPE's and using

$$G_s = \frac{1}{2\pi i}\oint \frac{dz}{z} z^{n+\frac{3}{2}} G(z) \tag{III.25}$$

we get

$$[L_n,L_m] = (n-m)L_{n+m} + \frac{D}{8}n(n^2-1)\delta_{n,-m}$$
$$[L_n,G_s] = (\frac{n}{2}-s)G_{n+s}$$
$$\{G_r,G_s\} = 2L_{r+s} + \frac{D}{2}(r^2-\frac{1}{4})\delta_{r,-s} \tag{III.26}$$

which is the same as in eq. (III.4) with $\hat{c}=D=\frac{2}{3}c$. ($\hat{c}$ is the super-Virasoro charge, while c is the central charge of a nonsupersymmetric Virasoro algebra. Each is normalized to take one unit for each dimension in their respective theories.)

The idea behind four-dimensional string theories is that the super-Virasoro operators T(z) and G(z) with $\hat{c}=D=10$ may be replaced by a sum of two terms:

$$T(z)\Big|_{D=10} = T(z)\Big|_{D=4} + T_{int}(z)\Big|_{\hat{c}=6}$$

$$G(z)\Big|_{D=10} = G(z)\Big|_{D=4} + G_{int}(z)\Big|_{\hat{c}=6}. \tag{III.27}$$

The first term on the right corresponds to a critical dimension D=4, and hence the name "four-dimensional strings," while the remaining terms carry the left-over $\hat{c}=6$, which must be included because the full theory still requires that

$\hat{c}_{total}=10$. In principle, the internal operators $T_{int}(z)$ and $G_{int}(z)$ can be represented by any consistent conformal field theory which has the right central charge, but clearly it is preferable to work with fields that can be defined in a Lagrangian and make physical sense. In the next chapter, I will discuss some four-dimensional string models that use free worldsheet fermions to define the internal operators $T_{int}(z)$ and $G_{int}(z)$. In addition to reducing the spacetime dimension to four, it will be shown that the free fermions can be used to represent a Kac-Moody algebra, and in this way we will introduce gauge symmetry into the theory.

## IV. Four-Dimensional Superstrings

In this section a new method of incorporating symmetry into superstring theory is used to compactify type II superstrings to four dimensions giving rise to dimension-18, semi-simple Lie groups.[2] There are only three such groups: $SU(2)^6$, $SU(3)\times SO(5)$, and $SU(4)\times SU(2)$ - the last two contain the standard gauge group.

When the FKS construction is used to incorporate a symmetry algebra G, the states at different mass levels of a single string together form level-one representations of the affine Kac-Moody algebra associated with G (at least before the left-right constraint is used to eliminate the tachyon). The highest weight states of level-one representations cannot transform according to the adjoint representation of G. In the heterotic string, the massless states in the adjoint representation, which provide the gauge vector bosons, are at the first excited level of the representation of the affine algebra, being obtained by the application of generators to the scalar tachyon (which itself decouples). For the spinning string, the massless states must themselves be highest-weight states of the affine algebra because the smaller mass gap means they cannot be reached from the tachyon by application of the generators. In order to obtain a representation of the affine algebra with adjoint representation highest-weight states, we need affine representations of level greater than one. In fact, such representations must have level number greater than or equal to $\hbar$, where $\hbar$ is the dual Coxeter number of the algebra $g$, which is n for SU(n) and n-2 for SO(n).[14,53]

The construction described in this section realizes a level-$\hbar$ representation of the affine algebra. In the FKS construction the number of compactified dimensions determines the rank of the gauge group, while here the dimension of the gauge group is determined by the number of internal $b_s$ or $d_n$ oscillators used in a fermionic description of the compactified degrees of freedom.

The low-energy limit of our four-dimensional string is N=4 supergravity coupled to N=4 supersymmetric Yang-Mills theory with the gauge group, whose algebra we have denoted by G, given by SU(3)×SO(5), SU(2)×SU(4), or $SU(2)^6$. The first case provides an example of a nonsimply laced group algebra realized in string theory. Further amendments to the model are necessary to reduce to one the number of residual supersymmetries and to obtain chiral fermions, but the model described here shows how to incorporate in superstring theories symmetry groups big enough to have the standard gauge group SU(3)×SU(2)×U(1) as a subgroup. Although the massless fermion content is not yet at this stage realistic, this method provides a mechanism by which the more economical type II superstring may be phenomenologically viable.

Fermionization and Generalizing the Gauge Group

In the original $SU(2)^6$ model, we start off with a spinning (type II) string in 10 dimensions and compactify it down to four dimensions while obtaining a representation of the gauge algebra in terms of the internal oscillators and vertex operators. In this section I will describe a four-dimensional string theory corresponding to the type II superstring that completely eliminates the extra dimensions by fermionizing them. The additional fermions that replace the bosonic coordinates as well as the

original fermionic partners of those coordinates are then used to represent the super-Virasoro and Kac-Moody algebras.[21,20] The mechanism thus simultaneously reduces the dimension of the theory at the same time that it introduces gauge symmetry.

The super-Virasoro generators of the 10-dimensional spinning string are given by:

$$\begin{aligned} T(z) &= \tfrac{1}{2}\alpha^i(z)\alpha^i(z) + \tfrac{1}{2}\partial_z b^i(z)b^i(z) \\ G(z) &= \alpha^i(z)b^i(z) \end{aligned} \tag{IV.1}$$

where the $\alpha$'s and b's obey eqs. (III.17) and (III.24). To fermionize this theory (dropping the i's for the moment), we let $b(z)=b^1(z)$ and we replace $\alpha(z)$ by two b's, $b^2(z)$ and $b^3(z)$, for each dimension we wish to eliminate (in this case six).

$$\begin{aligned} \alpha(z) &= -ib^2(z)b^3(z) \\ &= \frac{-i}{2} f_{1bc}b^b(z)b^c(z) \\ b,c &= 1,2,3. \end{aligned} \tag{IV.2}$$

Here $f_{1bc}$ are for the moment unspecified anti-symmetric constants. The supercurrent is then

$$G(z) = \frac{-i}{2}f_{1bc}b^1(z)b^b(z)b^c(z)$$

which may be put in the more symmetrical form:

$$G(z) = \frac{-i}{6}f_{abc}b^a(z)b^b(z)b^c(z). \tag{IV.3}$$

Using the same substitution in T(z), we get

$$\begin{aligned} T(z) &= \tfrac{1}{2}\alpha(z)\alpha(z) + \tfrac{1}{2}\partial_z b^1(z)b^1(z) \\ &= \tfrac{-1}{8} f_{1bc} f_{1b'c'} b^b(z)b^c(z)b^{b'}(z)b^{c'}(z) + \tfrac{1}{2}\partial_z b^1(z)b^1(z) \end{aligned}$$

and we may once again put this in a more symmetrical form by letting $1 \to a = 1,2,3$:

$$T(z) = \tfrac{-1}{8} f_{abc} f_{a'b'c'} b^b(z)b^c(z)b^{b'}(z)b^{c'}(z) + \tfrac{1}{2}\partial_z b^a(z)b^a(z).$$

The first term may be rewritten as

$$\begin{aligned} f_{abc} f_{ab'c'} b^b b^c b^{b'} b^{c'} &= \tfrac{1}{3}(f_{abc} f_{ab'c'} + f_{bb'a} f_{c'ca} + f_{bc'a} f_{cb'a}) b^b b^c b^{b'} b^{c'} \\ &= 0 \end{aligned}$$

if we require that the $f_{abc}$'s satisfy a Jacobi identity:

$$f_{abc} f_{ab'c'} + f_{bb'a} f_{c'ca} + f_{bc'a} f_{cb'a} = 0. \qquad \text{(IV.4)}$$

In this case the first term in T(z) vanishes, and the super-Virasoro generators are given by

$$\begin{aligned} T(z) &= \tfrac{1}{2}\partial_z b^a(z)b^a(z) \\ G(z) &= \tfrac{-i}{6} f_{abc} b^a(z)b^b(z)b^c(z). \end{aligned} \qquad \text{(IV.5)}$$

This eliminates one spacetime dimension when a=1,2,3. Since originally we had six compactified dimensions $\alpha^I$, $b^I$ for I=1,...,6, we must replace these by $\{b^{1I}, b^{2I}, b^{3I}\} = \{b^{aI}\}$ with a=1,2,3. We may represent the entire system of 18

fermions by $\{b^a(z)\}$ with a=1,...,18. Eq. (IV.5) with a defined this way represents the part of the super-Virasoro algebra corresponding to the six compactified dimensions (i.e., with $\hat{c}$=6). We may verify this explicitly by calculating the OPE's. We find:

$$
\begin{aligned}
T(z)T(z') &= \frac{\frac{1}{2}(\frac{d}{2})}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial_{z'}T(z')}{(z-z')} \\
G(z)G(z') &= \frac{\frac{1}{3}c_\psi(\frac{d}{2})}{(z-z')^3} + \frac{\frac{1}{2}c_\psi T(z')}{(z-z')} \\
T(z)G(z') &= \frac{\frac{3}{2}G(z')}{(z-z')^2} + \frac{\partial_{z'}G(z')}{(z-z')}.
\end{aligned}
\tag{IV.6}
$$

In deriving these OPE's, we have to again use the fact that the $f_{abc}$'s obey the Jacobi identity, eq. (IV.4). $c_\psi$ is defined by

$$
f_{abc}f_{dbc} = c_\psi \delta^{ad} \tag{IV.7}
$$

and d = # of fermions = 18. Using the normalization $c_\psi$=2 and setting $\hat{c}=\frac{d}{3}$, we get for the Fourier modes:

$$
\begin{aligned}
L_n &= \frac{1}{2}\sum_s (s-\frac{n}{2}) b^a_{n-s} b^a_s \\
G_s &= \frac{-i}{6} f_{abc} \sum_{s',n} b^a_{n-s'} b^b_{s'} b^c_{s-n}
\end{aligned}
$$

$$
[L_n, L_m] = (n-m)L_{n+m} + \frac{\hat{c}}{8} n(n^2-1)\delta_{n,-m}
$$

$$\{G_s,G_r\} = 2L_{r+s} + \frac{\hat{c}}{2}(r^2-\frac{1}{4})\delta_{r,-s}$$
$$[L_n,G_s] = (\frac{n}{2}-s)G_{n+s}. \qquad \text{(IV.8)}$$

For d=18, we get $\hat{c}$=6 as required. The remaining contribution comes from the four-dimensional spacetime components which are defined as in eq. (IV.1), but with $i=\hat{\imath}=1,2$.

Initially, $f_{abc}$ was defined as a set of arbitrary anti-symmetric constants with a,b,c=1,...,d. The consistency of the representation, eq. (IV.5), of the super-Virasoro algebra then required that $f_{abc}$ obey

$$f_{abc}f_{ab'c'} + f_{bb'a}f_{c'ca} + f_{bc'a}f_{cb'a} = 0$$

and

$$f_{abc}f_{dbc} = c_\psi\delta^{ad}. \qquad \text{(IV.9)}$$

But these are just the Jacobi identity and a normalization condition - precisely the requirements imposed on the structure constants of a Lie algebra. In fact, the only requirement we need to place on the generators in eq. (IV.5) is that the $f_{abc}$ be the structure constants of an arbitrary dimension-d, semi-simple Lie group. For d=18, there are only 3 possibilities: $SU(2)^6$, SO(5)×SU(3), and SU(2)×SU(4).

## Super-Kac-Moody Algebras

We may use the same set of fermions, $b^a(z)$ with a=1,...,d, to represent a Kac-Moody algebra:

$$T^a(z) = \frac{-i}{2}f_{abc}b^b(z)b^c(z). \qquad \text{(IV.10)}$$

Once again, to obtain the correct commutation relations, we need both of the conditions in eq. (IV.9), and we find:

$$T^a(z)T^b(z') = \frac{(\frac{c_\psi}{2})\delta^{ab}}{(z-z')^2} + if_{abc}\frac{T^c(z')}{(z-z')}. \tag{IV.11}$$

This is the operator product expansion of a Kac-Moody algebra corresponding to the dimension-d, semi-simple Lie group having $f_{abc}$ as its structure constants. The normalization is such that

$$k = (\frac{c_\psi}{2}) = 1. \tag{IV.12}$$

Since the representation level of the Kac-Mooy algebra is defined as

$$x = \frac{2k}{\psi^2}, \tag{IV.13}$$

and the dual Coxeter number is

$$\tilde{h} = \frac{c_\psi}{\psi^2}, \tag{IV.14}$$

eq. (IV.12) implies that we are working in a normalization in which

$$x = \tilde{h}, \tag{IV.15}$$

the level of the representation is given by the dual Coxeter number $\check{h}$, which is n for SU(n) and n-2 for SO(n). Hence we have a level (3,3) representation for SO(5)×SU(3), (2,4) for SU(2)×SU(4), and (2,2,2,2,2,2) for $SU(2)^6$.

Defining the modes as

$$T^a_n = \frac{1}{2\pi i}\oint \frac{dz}{z}\, z^{n+1} T^a(z) \tag{IV.16}$$

we get

$$[T^a_n, T^b_m] = if_{abc} T^c_{n+m} + n\delta_{n,-m}\delta^{ab}. \tag{IV.17}$$

The zero-graded elements of this generate our gauge symmetry.

The super-Virasoro and Kac-Moody algebras with generators $L_n$, $G_s$, and $T^a_n$ may be combined into a single closed algebra, refered to here as a super-Kac-Moody algebra, by including the free worldsheet fermions $b^a_s$ in the algebra.[54] The complete algebra then has three parts: the super-Virasoro algebra, with generators $(L_n, G_s)$ having conformal spins 2 and $\frac{3}{2}$, respectively; the (super) Kac-Moody algebra $(T^a_n, b^a_s)$ whose generators have spin 1 and $\frac{1}{2}$; and the mixing between the two algebras:

super-Virasoro

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{\hat{c}}{8}n(n^2-1)\delta_{n,-m} \\ \{G_s, G_r\} &= 2L_{r+s} + \frac{\hat{c}}{2}(r^2-\frac{1}{4})\delta_{r,-s} \\ [L_n, G_s] &= (\frac{n}{2}-s)G_{n+s} \end{aligned} \tag{IV.18}$$

(super) Kac-Moody

$$\begin{aligned} [T^a_n, T^b_m] &= if_{abc}T^c_{n+m} + n\delta_{n,-m}\delta^{ab} \\ [T^a_n, b^b_s] &= if_{abc}b^c_{n+s} \\ \{b^a_s, b^b_r\} &= \delta_{s,-r}\delta^{ab} \end{aligned} \tag{IV.19}$$

mixing

$$\begin{aligned}
[L_n, T^a_m] &= -mT^a_{n+m} \\
[L_n, b^a_s] &= -(\tfrac{n}{2}+s)b^a_{n+s} \\
[G_s, T^a_n] &= -nb^a_{s+n} \\
\{G_s, b^a_r\} &= T^a_{r+s}.
\end{aligned} \tag{IV.20}$$

The spectrum of the model presented in this section using free fermions to represent the gauge algebra falls into representations of this full super-Kac-Moody algebra.

The Ramond sector obeys a similar algebra, but with $F_n$ and $d^a_n$ operators instead of the NS operators $G_s$ and $b^a_s$. The Ramond oscillators are integrally moded, corresponding to the fact that they obey periodic boundary conditions. In terms of conformal fields, both the R and NS fermions are represented as spin-$\frac{1}{2}$ fermions obeying $\psi^a(z)\psi^b(z') = \frac{\delta^{ab}}{(z-z')}$. The difference between the two theories stems from how the physical states are defined.[19] The NS physical ground state is obtained from the SL(2,R) invariant vacuum $|0\rangle$, the vacuum of the conformal theory, by acting on it with the NS tachyon vertex operator $V_0(z)$ in the limit $z\to 0$ (corresponding to $\tau\to-\infty$, for an "in" state). The Ramond ground state is created by acting on the vacuum with a spin operator $\Sigma(z)$:

$$|0\rangle_R = \Sigma(0)|0\rangle \quad .$$

The OPE of $\Sigma(z)$ with the fermion operators contains square roots. The spin operator is thus nonlocal with respect to the fermions. The other states in the Ramond sector are built by acting on the Ramond ground state with superconformal fields.

## N=2 Superconformal Algebras

A question which arose during the course of this work was whether it is possible to define a combined super-Virasoro and Kac-Moody algebra for the N=2 superconformal theory. Such an N=2 super-Kac-Moody algebra could conceivably be used as a string compactification by adding enough of them together to form the $\hat{c}=6$ part of the compactified superstring. The idea here is that any compactified string theory is given in terms of some conformal field theory and that any conformal field theory is a consistent string theory provided the central charges add up correctly. So even though it might be hard to define a sensible Lagrangian in four dimensions for compactified theories consisting of a sum of exotic conformal theories, it is still a worthwhile endeavor to examine all possible conformal theories that could contribute the right central charge. Also, it has recently been shown[57,60] that (0,2) worldsheet supersymmetry ensures the existence of N=1 spacetime supersymmetry. Thus N=2 superconformal invariance is an essential ingredient in realistic models.

The idea of using N=2 superconformal theories as a compactification of the heterotic theory has been studied by Gepner.[55] He obtained modular-invariant, four-dimensional models with gauge groups which contain $E_6$ with chiral fermions. The gauge groups come from the bosonic FKS construction used in the heterotic string; realistic gauge groups are not obtained by the new method of compactification alone.

What I decided to look for was a way of representing the N=2 superconformal generators in terms of free fermions in a way that that could be combined with a representation of a Kac-Moody algebra also in terms of the

free fermions. Such a combined N=2 super-Virasoro and Kac-Moody algebra would then be the N=2 analog of the Kac-Todorov N=1 super-Kac-Moody algebra presented in the previous section.[54] The hope was to find such a representation because then it could be used not only as a compactification but also as a new way of introducing symmetry into the theory. I was not able to find such a combined representation in terms of free fermions, however. The N=2 super-Virasoro algebra has generators $(L_n, G_s, \bar{G}_s, \Gamma_n)$ with conformal spins $(2, \frac{3}{2}, \frac{3}{2}, 1)$ and can be represented in terms of complex fermions $b^a_s$ and $\bar{b}^a_s$. Here $\Gamma_n$ is a U(1) generator which links $G_s$ and $\bar{G}_s$. The complex fermions can also be used to create Kac-Moody generators $T^a_n$, but the mixing of these operators with the super-Virasoro operators generates new operators which are not in the combined set, so we do not get a closed algebra.

The N=2 superconformal generators are given in terms of complex fermions by[56-57]

$$\begin{aligned}
T(z) &= \frac{-1}{12} f_{abc} f_{ab'c'} b^b(z) b^c(z) \bar{b}^{b'}(z) \bar{b}^{c'}(z) \\
&\qquad + \frac{1}{6}(\partial_z b^a(z) \bar{b}^a(z) + \partial_z \bar{b}^a(z) b^a(z)) \\
G(z) &= \frac{-i}{3\sqrt{6}} f_{abc} b^a(z) b^b(z) b^c(z) \\
\bar{G}(z) &= \frac{-i}{3\sqrt{6}} f_{abc} \bar{b}^a(z) \bar{b}^b(z) \bar{b}^c(z) \\
\Gamma(z) &= \frac{1}{3} b^a(z) \bar{b}^a(z)
\end{aligned} \tag{IV.21}$$

where a=1,2,...d, and

$$\begin{aligned}
\langle b^a(z) \bar{b}^b(z') \rangle &= \frac{\delta^{ab}}{(z-z')} \\
\langle b^a(z) b^b(z') \rangle &= \langle \bar{b}^a(z) \bar{b}^b(z') \rangle = 0.
\end{aligned} \tag{IV.22}$$

The first term in $T(z)$ does not vanish in this case because $b$ and $\bar{b}$ are not indistinguishable, so the Jacobi identity cannot be used here to eliminate the first term. The normalization of the structure constants is such that $c_\psi=2$. From the OPE's one gets the N=2 super-Virasoro algebra:

$$
\begin{aligned}
[L_n,L_m] &= (n-m)L_{n+m} + \frac{(\frac{d}{3})}{12}(n^3-n^2)\delta_{n,-m} \\
\{G_r,G_s\} &= \{\bar{G}_r,\bar{G}_s\} = 0 \\
\{G_r,\bar{G}_s\} &= 2L_{r+s} + (r-s)\Gamma_{r+s} + \frac{1}{3}(\frac{d}{3})(r^2-\frac{1}{4})\delta_{r,-s} \\
[L_n,G_s] &= (\frac{n}{2}-s)G_{n+s} \\
[L_n,\bar{G}_s] &= (\frac{n}{2}-s)\bar{G}_{n+s} \\
[\Gamma_n,\Gamma_m] &= \frac{1}{3}n(\frac{d}{3})\delta_{n,-m} \\
[L_n,\Gamma_m] &= -m\Gamma_{n+m} \\
[\Gamma_n,G_r] &= G_{n+r} \\
[\Gamma_n,\bar{G}_r] &= \bar{G}_{n+r}.
\end{aligned}
\qquad \text{(IV.23)}
$$

This is the same as Gepner's algebra with $c=\frac{d}{3}$.[55]

The N=2 algebra has a dicrete series of critical values as do the N=1 and N=0 Virasoro algebras:[13,52,56,57]

$$
\begin{array}{lll}
N=2: & c = \dfrac{3k}{k+2} & k = 1,2,3,\ldots \\
N=1: & \hat{c} = \frac{2}{3}c = 1 - \dfrac{8}{k(k+2)} & k = 0,1,2,\ldots \\
N=0: & c = 1 - \dfrac{6}{(k+2)(k+3)} & k = 2,3,4,\ldots
\end{array}
$$

A Kac-Moody algebra can be formed out of the complex $b$'s by:

$$T^a(z) = -if_{abc}b^b(z)\bar{b}^c(z) \qquad \text{(IV.24)}$$

$$\{b^a_r, b^b_s\} = \delta_{r,-s}\delta^{ab}$$
$$\{b^a_r, b^b_s\} = \{\bar{b}^a_r, \bar{b}^b_s\} = 0 \qquad \text{(IV.25)}$$

$$[T^a_n, T^b_m] = if_{abc}T^c_{n+m} + 2n\delta_{n,-m}\delta^{ab}. \qquad \text{(IV.26)}$$

But if one tries to form a combined N=2 super-Virasoro and Kac-Moody algebra, the algebra will not close because of the mixing terms. For example:

$$\{G_r, b^a_s\} \neq T^a_{r+s}$$

and

$$[G_r, T^a_n] \neq b^a_{n+r}.$$

All of the various ways of representing $T^a_n$ in terms of free fermions do not mix properly with the super-Virasoro algebra to give a closed combined algebra. In a sense, the N=2 superconformal algebra is already a combined super-Virasoro and Kac-Moody algebra because it contains the U(1) operators $\Gamma_n$ (which represent an abelian Kac-Moody algebra). Perhaps this is the largest algebra that the theory can contain.

## The SO(5)×SU(3) and SU(4)×SU(2) Models

We obtain the new models by replacing the internal bosons $\alpha^I_n$ by a pair of fermions $b^I_{(2)_r}$ and $b^I_{(3)_r}$

$$\alpha^I_n = -i\sum_s b^I_{(2)_{n-s}} b^I_{(3)_s} \qquad \text{(no sum on I).} \qquad \text{(IV.27)}$$

These together with the internal fermions $b^I_s$ form a set of 18 fermions that may be denoted collectively as $b^a_s$, a=1,...,18. The spectrums of states in the

compactified models are the same as can be seen from the partition functions corresponding to the internal degrees of freedom:

$$\prod_{n=1}^{\infty} (1-w^n)^{-6} \prod_{s=1/2}^{\infty} (1+w^s)^6 \, [\theta_3(0|\tau)]^6 = \prod_{s=1/2}^{\infty} (1+w^s)^{18}. \qquad \text{(IV.28)}$$

The three terms on the left count the $\alpha_n^I$, $b_s^I$, and $|p^I\rangle$ states, respectively; this is identically equal to the number of states generated by the 18 $b_s^a$ fermions whose partition function is on the right-hand side.

The Kac-Moody generators are given in terms of the b-oscillators by

$$T_n^a = \frac{-i}{2} f_{abc} \sum_{s=-\infty}^{\infty} b_{n-s}^b b_s^c \qquad \text{(IV.29)}$$

where the zero modes are the gauge generators. The $f_{abc}$'s are the structure constants of any dimension-18, semi-simple Lie algebra. This results in two new models besides the $SU(2)^6$ model with the new gauge groups $SO(5)\times SU(3)$ and $SU(4)\times SU(2)$. These are rank-4 gauge groups, both of which are large enough to contain the standard model.

The mass operator in the new formalism is

$$\frac{1}{2}\alpha' m_4^2 = N + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^{\hat{I}} \tilde{\alpha}_n^{\hat{I}} + \sum_{s=1/2}^{\infty} s b_{-s}^{\hat{I}} b_s^{\hat{I}} + \sum_{s=1/2}^{\infty} s b_{-s}^a b_s^a + \alpha' (\bar{p}^I)^2 - \frac{1}{2} \qquad \text{(IV.30)}$$

and the left-right constraint is

$$N + \alpha'(\bar{p}^I)^2 = \sum_{n=1}^{\infty} \tilde{\alpha}^{\hat{i}}_{-n}\tilde{\alpha}^{\hat{i}}_{n} + \sum_{s=1/2}^{\infty} s b^{\hat{i}}_{-s} b^{\hat{i}}_{s} + \sum_{s=1/2}^{\infty} s b^{a}_{-s} b^{a}_{s} - \frac{1}{2} \quad . \tag{IV.31}$$

In addition to the N=4 supergravity multiplet, the massless states are $\{|\hat{i}\rangle_R, |I\rangle_R, |a\rangle_R\} \times \{b^a_{-1/2}|0\rangle_L\}$, the d=4, N=4 supersymmetric Yang-Mills multiplet in any dimension-18, semi-simple Lie group.

## Tree and One-Loop Amplitudes

For the generalized dimension-18 groups, the vertex operators for $|i\rangle_R \times b^a_{-\frac{1}{2}}|0\rangle_L$ which replace eq. (II.24) are

$$W^{ia}(k^{\hat{i}},z,\sigma) = g\Big(\frac{1}{\sqrt{2\alpha'}} \sum_n \alpha^i_n e^{-2in(\tau-\sigma)} + \frac{1}{2}k^{\hat{k}} R^{i\hat{k}}\Big)\, e^{ik^{\hat{i}}X^{\hat{i}}(\sigma,\tau)} \cdot \Big[\frac{-i}{2\sqrt{2\alpha'}} f_{abc} H^b(z) H^c(z) - H^a(z)\frac{1}{2}k^{\hat{i}} H^{\hat{i}}(z)\Big] \;, \tag{IV.32}$$

where $H^a(z) = \sum_{s=-\infty}^{\infty} b^a_s e^{-2in(\tau+\sigma)}$. The three-point gauge coupling is the amplitude involving three states $|\hat{i}\rangle_R \times b^a_{-1/2}|0\rangle_L$. In tree approximation,

$$\begin{aligned} A^{abc}(k^{\hat{k}}_1,k^{\hat{i}}_2,k^{\hat{1}}_3) &= \langle \epsilon_1, k^{\hat{k}}_1| b^a_{\frac{1}{2}} \frac{1}{\pi}\epsilon^{\hat{i}}_2 \int_0^{\pi} d\sigma\, W^{\hat{i}b}(k^{\hat{i}}_2,1,\sigma)\, b^a_{-\frac{1}{2}}|\epsilon_3, k^{\hat{1}}_3\rangle \\ &= if_{abc} g \epsilon^{\mu}_1 \epsilon^{\nu}_2 \epsilon^{\zeta}_3 [(k_2-k_1)^{\mu}\delta^{\nu\zeta} + (k_3-k_2)^{\nu}\delta^{\mu\zeta} + (k_1-k_3)^{\zeta}\delta^{\mu\nu}] \;. \end{aligned} \tag{IV.33}$$

Eq. (IV.33) is the standard Yang-Mills three-point coupling and proves that the affine algebraic compactification gives rise to standard interacting nonabelian gauge bosons.

The propagator becomes

$$\Delta_c = \frac{\alpha'}{2\pi}\int_D d^2z \; z^{N + \alpha'(\bar{p}^I)^2 \sum_{n=1}^{\infty} \tilde{\alpha}^{\hat{i}}_{-n}\tilde{\alpha}^{\hat{i}}_{n} + \sum_{s=1/2}^{\infty} s b^{\hat{i}}_{-s} b^{\hat{i}}_{s} + \sum_{s=1/2}^{\infty} s b^{a}_{-s} b^{a}_{s} - \frac{1}{2}} \; |z|^{\frac{\alpha' p^2}{2} - 2} \; . \tag{IV.34}$$

The one-loop amplitude for external vector mesons is

$$\begin{aligned} A^{abcd}_{loop}(1234) &= \epsilon^{\hat{i}}_1 \epsilon^{\hat{j}}_2 \epsilon^{\hat{k}}_3 \epsilon^{\hat{l}}_4 \int dp \; \mathrm{tr}(\Delta_c W^{\hat{i}a}(1)\Delta_c W^{\hat{j}b}(2)\Delta_c W^{\hat{k}c}(3)\Delta_c W^{\hat{l}d}(4)) \\ &= \left(\frac{\alpha' g}{2\pi}\right)^4 \frac{1}{16} K \frac{(2\pi)^8}{(\alpha')^2} \int_{F_{SM}} d^2\tau \; (\mathrm{Im}\;\tau)^{-2} g^{abcd}(\tau) \quad , \end{aligned} \tag{IV.35}$$

where the integrand is again modular invariant when we rewrite it in the form:

$$\int_{F_{SM}} d^2\tau \; (\mathrm{Im}\;\tau)^{-2} g^{abcd}(\tau) = \int_F d^2\tau \; (\mathrm{Im}\;\tau)^{-2}[g^{abcd}(\tau) + g^{abcd}(\tau+1) + g^{abcd}(-\tfrac{1}{\tau}+1)] . \tag{IV.36}$$

From eq. (IV.35), we find

$$\int d^2\tau (\mathrm{Im}\ \tau)^{-2} g^{abcd}(\tau) = \int d^2\tau \left(\frac{-2\pi}{\ln|w|}\right)^2 \int \prod_{I=1}^{3} d^2\nu_I \prod_{I<J} (\chi_{IJ})^{\alpha' \hat{k}_I \hat{k}_J} (f(\bar{w}))^{-2}$$
$$\cdot \bar{w}^{-\frac{1}{2}} \prod_{s=1/2}^{\infty} (1+\bar{w}^s)^{20} \sum_{\sqrt{2\alpha'}\bar{p}^I \in Z^6} w^{\alpha'(\bar{p}^I)^2} a^{abcd}(X^+) \qquad \text{(IV.37)}$$

where $a^{abcd}(X)$ is given by

$$\begin{aligned} a^{IJKL}(X) = \frac{1}{16}\left(\frac{1}{2\alpha'}\right)^2 [& 4[g_{IJ}g_{KL}X_{43}^2X_{21}^2 + g_{IK}g_{JL}X_{42}^2X_{31}^2 + g_{IL}g_{JK}X_{41}^2X_{32}^2] \\ & + 8[(T_{IJKL}+T_{ILKJ})X_{43}X_{41}X_{32}X_{21} - (T_{IKJL}+T_{ILJK})X_{42}X_{41}X_{32}X_{31} \\ & \qquad - (T_{IJLK}+T_{IKLJ})X_{43}X_{42}X_{31}X_{21}] \\ & + (2\alpha')[4\{(sM_{ILJK} - uM_{IJKL})X_{43}X_{41}X_{32}X_{21} \\ & \qquad + (tM_{ILJK} + uM_{IKJL})X_{42}X_{41}X_{32}X_{31} \\ & \qquad + (-sM_{IKJL} - tM_{IJKL})X_{43}X_{42}X_{31}X_{21}\} \\ & \quad + s(\delta_{IJ}g_{KL} + g_{IJ}\delta_{KL})X_{43}^2X_{21}^2 \\ & \quad + t(\delta_{JL}g_{IK} + g_{JL}\delta_{IK})X_{21}^2X_{31}^2 \\ & \quad + u(\delta_{IL}g_{JK} + g_{IL}\delta_{JK})X_{41}^2X_{32}^2] \\ & + (2\alpha')^2 \frac{1}{4}(s^2X_{43}X_{21} - t^2X_{42}X_{31} + u^2X_{41}X_{32}) \\ & \cdot(\delta_{IJ}\delta_{KL}X_{43}X_{21} - \delta_{IK}\delta_{JL}X_{42}X_{31} + \delta_{IL}\delta_{JK}X_{41}X_{32})] \end{aligned} \qquad \text{(IV.38)}$$

where
$$g_{IJ} = f_{IPQ}f_{JPQ}$$
$$T_{IJKL} = f_{IPQ}f_{JQR}f_{KRS}f_{LSP}$$
$$M_{IJKL} = f_{IJR}f_{KLR}.$$

Using the transformation properties of the Jacobi theta functions, we find

$$\int d^2\tau (\mathrm{Im}\ \tau)^{-2} g^{abcd}(\tau+1) = \int d^2\tau \left(\frac{-2\pi}{\ln|w|}\right)^2 \int \prod_{I=1}^{3} d^2\nu_I \prod_{I<J} (\chi_{IJ})^{\alpha' k_I^{\hat{i}} k_J^{\hat{i}}} (f(\bar{w}))^{-2}$$
$$(-\bar{w})^{-\frac{1}{2}} \prod_{s=1/2}^{\infty} (1-\bar{w}^s)^{20} \sum_{\sqrt{2\alpha'}\bar{p}^I \in Z^6} w^{\alpha'(\bar{p}^I)^2} a^{abcd}(\tilde{X}^+) \qquad (IV.39)$$

and

$$\int d^2\tau (\mathrm{Im}\ \tau)^{-2} g^{abcd}(-\tfrac{1}{\tau}+1) = \int d^2\tau \left(\frac{-2\pi}{\ln|w|}\right)^2 \int \prod_{I=1}^{3} d^2\nu_I \prod_{I<J} (\chi_{IJ})^{\alpha' k_I^{\hat{i}} k_J^{\hat{i}}} (f(\bar{w}))^{-2}$$
$$2^{10}(-\bar{w})^{\frac{3}{4}} \prod_{n=1}^{\infty} (1+\bar{w}^n)^{20} \sum_{\sqrt{2\alpha'}\bar{p}^I \in (Z+\frac{1}{2})^6} w^{\alpha'(\bar{p}^I)^2} a^{abcd}(\tilde{X}^+) \ . \qquad (IV.40)$$

We identify the contribution of eqs. (IV.37) and (IV.39) as given by

$$\left(\frac{\alpha' g}{2\pi}\right)^4 \frac{1}{16\alpha'^2} K\ (2\pi)^8 \int_F d^2\tau (\mathrm{Im}\ \tau)^{-2} \left(\frac{g(\tau) + g(\tau+1)}{2}\right)^{abcd}$$
$$= \epsilon_1^{\hat{i}} \epsilon_2^{\hat{j}} \epsilon_3^{\hat{k}} \epsilon_4^{\hat{l}} \int dp\ \mathrm{tr}\left(\frac{1+(-1)^P}{2}\right) \Delta_c W^{\hat{i}a}(1) \left(\frac{1+(-1)^P}{2}\right) \Delta_c W^{\hat{j}b}(2)$$
$$\cdot \left(\frac{1+(-1)^P}{2}\right) \Delta_c W^{\hat{k}c}(3) \left(\frac{1+(-1)^P}{2}\right) \Delta_c W^{\hat{l}d}(4) \qquad (IV.41)$$

where $\qquad P = \sum_{s=1/2}^{\infty} b_{-s}^a b_s^a - 1 + 2\alpha'(\bar{p}^I)^2 + \sum_{s=1/2}^{\infty} b_{-s}^{\hat{i}} b_s^{\hat{i}}.$

Similarly eq. (IV.40) is identified as

$$(\frac{\alpha' g}{2\pi})^4 \frac{1}{16\alpha'^2} K (2\pi)^8 \int_F d^2\tau (\mathrm{Im}\ \tau)^{-2} (\frac{g^{abcd}(\frac{-1}{\tau}+1)}{2})$$

$$= -\epsilon_1^{\hat{\imath}}\epsilon_2^{\hat{\jmath}}\epsilon_3^{\hat{k}}\epsilon_4^{\hat{l}} \frac{1}{2} \int \ d p\,\mathrm{tr} (\frac{1+(-1)^{\bar{P}}}{2}) \bar{\Delta}_c \bar{W}^{\hat{\imath}a}(1) (\frac{1+(-1)^{\bar{P}}}{2})\ \bar{\Delta}_c \bar{W}^{\hat{\jmath}b}(2)$$

$$\cdot (\frac{1+(-1)^{\bar{P}}}{2})\ \bar{\Delta}_c \bar{W}^{\hat{k}c}(3) (\frac{1+(-1)^{\bar{P}}}{2}) \bar{\Delta}_c \bar{W}^{\hat{l}d}(4) \qquad \text{(IV.42)}$$

where

$$(-1)^{\bar{P}} = 2^{10} d_o^1 d_o^2 \cdots d_o^{18} d_o^{\hat{1}} d_o^{\hat{2}} (-1)^{\sum_{n=1}^{\infty} d_{-n}^a d_n^a + 2\alpha'(\bar{p}^I)^2 + \sum_{n=1}^{\infty} d_{-n}^{\hat{\imath}} d_n^{\hat{\imath}}} .$$

$\bar{W}^{\hat{\imath}a}(k^{\hat{\jmath}})$ is the vertex emission operator for vector bosons $|\hat{\imath}\rangle_R \times b^a_{-1/2}|0\rangle_L$ from a fermion line,[45] and $\bar{\Delta}_c$ is calculated from the mass operator for a massive SS×R sector where the right movers have been compactified on a shifted $(Z+\frac{1}{2})^6$ lattice:

$$\bar{W}^{\hat{\imath}a}(k^{\hat{\imath}},z,\sigma) = g(\frac{1}{\sqrt{2\alpha'}} \sum_n \alpha_n^{\hat{\imath}} e^{-2in(\tau-\sigma)} + \frac{1}{2} k^{\hat{k}} R^{\hat{\imath}\hat{k}})\ e^{ik^{\hat{\imath}} X^{\hat{\imath}}(\sigma,\tau)}$$

$$\cdot (\frac{-i}{2} \frac{1}{\sqrt{2\alpha'}} f_{abc} \frac{\Gamma^b(z)}{i\sqrt{2}} \frac{\Gamma^c(z)}{i\sqrt{2}} - \frac{k^{\hat{l}}}{2} \frac{\Gamma^a(z)}{i\sqrt{2}} \frac{\Gamma^{\hat{l}}(z)}{i\sqrt{2}}) \qquad \text{(IV.43)}$$

where

$$\Gamma^a(z) = \gamma^a + i\sqrt{2}\gamma_{21} \sum_{n\neq 1} d_n^a e^{-2in(\tau+\sigma)}, \qquad \gamma_{21} = 2^{10} d_o^1 d_o^2 \cdots d_o^{18} d_o^{\hat{1}} d_o^{\hat{2}},$$

$$\Gamma^{\hat{\imath}}(z) = \gamma^{\hat{\imath}} + i\sqrt{2}\gamma_{21} \sum_{n\neq 1} d_n^{\hat{\imath}} e^{-2in(\tau+\sigma)}, \qquad \Gamma_3 = \gamma_3 (-1)^{\sum_{n=1}^{\infty} d_{-n}^{\hat{\imath}} d_n^{\hat{\imath}}},$$

$$\bar{\Delta}_c = \frac{\alpha'}{2\pi}\int_D d^2z\; z^{\bar{N} + \alpha'(\bar{p}^I)^2 \sum_{n=1}^{\infty}\tilde{\alpha}^{\hat{\imath}}_{-n}\tilde{\alpha}^{\hat{\imath}}_{n} + \sum_{n=1}^{\infty} n d^{\hat{\imath}}_{-n} d^{\hat{\imath}}_{n} + \sum_{n=1}^{\infty} n d^{a}_{-n} d^{a}_{n} + \frac{3}{4}}\; \bar{z}\; |z|^{\frac{\alpha' p^2}{2} - 2} \; . \tag{IV.44}$$

The lowest state in the SS×R sector is $e^{i\hat{x}^I \bar{p}^I}\{|\hat{\imath}\rangle_R, |I\rangle_R, |a\rangle_R\}\times\{|\bar{a}\rangle_L\}$ where $|\bar{a}\rangle_L$ is the dimension $2^9$ spinor ground state of the internal Ramond modes.

I have described here a compactification technique which is useful in reducing a string theory from ten to four dimensions, and which, for theories with no gauge group in ten dimensions, inserts a nonabelian gauge symmetry in four dimensions. The motivation for this is that other compactifications to four dimensions either start with a gauge group in ten dimensions or pick up the gauge group via compactification from twenty-six dimensions, so that before symmetry breaking these give rise to rank 22=26-4 gauge groups. The compactification procedure for type II superstrings described here gives rise to gauge groups with rank less than or equal to 6=10-4, and thus appears to be a more economical model for low-energy phenomenology. Indeed, this mechanism, although it does not explain independently why we live in four dimensions, has, with its new way of introducing symmetry, exactly matched the fact that we live in four dimensions with the size of the symmetry group which governs four-dimensional interactions.

## V. Unitarity and Modular Invariance of Four-Dimensional Strings

In the previous sections it was shown that compactification of the type II superstring using free worldsheet fermions to represent the internal degrees of freedom can produce a four-dimensional string model with an arbitrary 18-dimensional semi-simple Lie group - the possibilities being $SU(2)^6$, $SU(3)\times SO(5)$, or $SU(2)\times SU(4)$. The last two examples contain the gauge group of the standard model, $SU(3)\times SU(2)\times U(1)$, and it is possible using similar methods to derive a four-dimensional string theory where the symmetry is reduced to this subgroup. Although the massless spacetime fermion content of these models does not yet completely match the physically observed fermion representations,[33,59] the natural appearance of rank 4 gauge groups is appealing and suggests that a deeper understanding of these superstring theories, including supersymmetry breaking, may lead unambiguously to a model with a realistic low energy spectrum and thereby establish contact with experiment.

In this section, which describes the results of ref. [3], we pursue the study of such models further, and we generalize to the case of twisted fermions, which obey arbitrary boundary conditions - not just periodic (R) or anti-periodic (NS). First, for theories whose internal degrees of freedom are described by free fermions, twisted or untwisted, we derive a set of consistency conditions directly from the requirements of 1) making a consistent set of projections defined by fermionic number operators, which generalize the GSO projection, and 2) the modular invariance of the one-loop partition function. Secondly, a simple "box" notation is introduced, and

these diagrams are used to display a number of interesting models exemplifying features we expect to survive in a realistic type II string theory containing the standard model.

Previous approaches to the classification of models in string theory have exploited the modular invariance of multi-loop amplitudes in the derivation of consistency conditions.[34,35,60] We show that within the above framework, the conditions others have deduced from multi-loop modular invariance are in fact consequences of the requirements of unitarity at the tree diagram and one-loop levels. In particular, we identify the conditions coming from the conservation at vertices of the quantum numbers associated with the projections. Even though we make use of conditions already identified in refs. [35,60], we organize the argument differently and take projections in the various sectors to be the primary concept, whereas consideration of multi-loop modular invariance concentrates on an approach oriented towards the worldsheet properties of the amplitudes. Since the real guide we have in the construction of string theories at present is the requirement of unitarity, as applied in a formal sense to the perturbation series, it is useful to have a discussion based directly on unitarity. Indeed, string theory had its origins in the requirement that the Veneziano amplitude be consistent with unitarity in this way, and the later construction of amplitudes describing the interactions of fermions was based on it.

Working in a notation similar to that introduced by Antoniadis, Bachas, and Kounnas,[35] we derive, for models that are described in terms of free fermions and that are unitary on a set of states we pick out from the whole Fock space using projection operators, conditions for (A) the set of projections in each sector to be consistent, and for (B) the quantum numbers associated with these projections to be conserved at each vertex. Condition

(A) follows from the definition of the projection operators and condition (B) is needed for unitarity at the tree level. Next we give the conditions for (C) the modular invariance of one-loop integrands. Conditions (A) and (C) are sufficient to imply the conditions of multi-loop modular invariance given in ref. [35]. Although our discussion of consistency conditions is in the context of type II theories, describable in terms of free fermion fields, it is also applicable to heterotic models and seems likely to be general. We will also extend the analysis to the case of twisted fermions in order to describe models with chiral spacetime fermions.

We have formulated the consistency conditions for four-dimensional strings in as simple a form as possible to enable a clear discussion of specific cases to be given. In particular, we see that the SU(3)×U(1) quantum numbers of the left-handed SU(2) quark doublet $(3,\frac{1}{3})$ and the neutrino-electron doublet (1,-1) arise naturally as the spinor representation of a twisted affine SU(4) algebra in a version of the above four-dimensional string theories, albeit without an SU(2) algebra.

Untwisted Fermions

We shall start by considering the fermionic formulation using untwisted fermions - these obey either periodic (R) or anti-periodic (NS) boundary conditions. The type II string in the light-cone description, compactified to four-dimensional spacetime, then has all its internal degrees of freedom described by fermionic fields. The left- and right-moving modes are each taken to be described by 2 bosonic and 20 fermionic fields: $\tilde{\alpha}^{\hat{\imath}}_n$, $\tilde{h}^{\hat{\imath}}_r$, $\tilde{h}^a_r$; $\alpha^{\hat{\imath}}_n$, $h^{\hat{\imath}}_r$, $h^a_r$; $\hat{\imath}$ = 1,2; 1 ≦ a ≦ 18, where the superscript $\hat{\imath}$ refers to the transverse spatial degrees of freedom and the supersript a to the internal ones. We

denote the fermi fields generically by $h^{\mu}_{r}$, $1 \leqq \mu \leqq 40$. Each sector consisting of these fermions will be represented pictorially by a set of boxes with 20 fermions on the left and 20 on the right. The division of a 20 + 20 box into subboxes will represent the groupings of the fermions used in the projections on that sector, and the shadings of the boxes will denote the boundary conditions - either shaded (R) or unshaded (NS).

The one-loop contribution to the vacuum-to-vacuum amplitude, or the cosmological constant, is

$$\Lambda = \frac{-1}{4\pi(\alpha')^2}\int_F d^2\tau(\mathrm{Im}\ \tau)^{-3}|f(w)|^{-4}\Lambda_f \tag{V.1}$$

where $\Lambda_f$ is the partition function for the fermionic degrees of freedom,

$$\Lambda_f = \sum_{\alpha\varepsilon\Omega}\delta_\alpha \mathrm{tr}_\alpha\{\bar{w}^{\tilde{L}_0-\frac{1}{2}}\, w^{L_0-\frac{1}{2}}\prod_{\beta\varepsilon\Omega}P_{\alpha,\beta}\}, \tag{V.2}$$

i.e. the spectrum of a theory will consist of a number of sectors, charcterized by which of the $h^{\mu}_{r}$ are Ramond ($r \varepsilon Z$; periodic), the rest being Neveu-Schwarz ($r \varepsilon Z + \frac{1}{2}$; anti-periodic). The quantities $\delta_\alpha$ and the projection operators $P_{\alpha,\beta}$ are discussed below. The integration region $F=\{\tau:|\tau|>1, |\mathrm{Re}\tau|<\frac{1}{2}\}$ is a fundamental region for the modular group.

A sector can thus be labelled by the subset $\alpha$ of all the Fermi fields,

$$F = \{h^{\mu}_{r}:\ 1 \leqq \mu \leqq 40\}, \tag{V.3}$$

which are periodic in that sector. In the sector $\alpha$, $h^\mu$ is Ramond if $h^\mu \varepsilon \alpha$ and $h^\mu$ is Neveu-Schwarz if $h^\mu \not\varepsilon \alpha$, i.e.

$$h^\mu \varepsilon \bar{\alpha} = \{h^\mu \varepsilon F : h^\mu \not\varepsilon \alpha\}, \tag{V.4}$$

the complement of $h^\mu$ in F. Thus the sector in which all the $h^\mu$ are NS is denoted by the empty set $\emptyset$ and that in which all the $h^\mu$ are R is denoted by F.

We assume that the set of states on which the theory is unitary will be specified by states that survive certain projections defined by number operators. (These generalize the GSO projection.) The projections are defined by requiring the parity of the number operators, $N_\beta$, which when acting on a sector $\alpha$ are realized in terms of the fermi fields in $\alpha$, to take on definite values $\varepsilon(\alpha,\beta)$ on any state in the sector $\alpha$, i.e.

$$(-1)^{N_\beta}\Big|_\alpha = \varepsilon(\alpha,\beta), \tag{V.5}$$

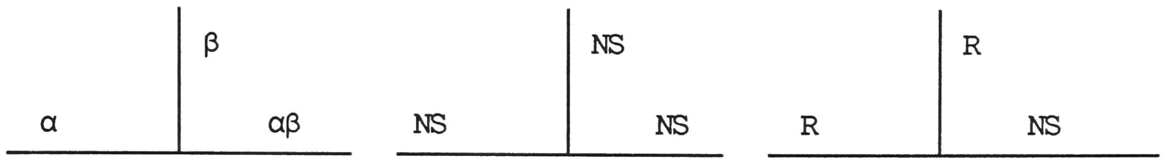
where each $\varepsilon(\alpha,\beta)$ is either $\pm 1$. The spectrum of a model is thus completely specified by a collection $\Omega$ of subsets of F, which label the sectors, together with a set $\{(-1)^{N_\beta} : \beta \varepsilon \Omega'\}$ of parity operators, and their values $\varepsilon(\alpha,\beta)$ on the sectors $\alpha \varepsilon \Omega$. In fact, we shall see that modular invariance implies $\Omega=\Omega'$.

If $\alpha,\beta$ correspond to arbitrary sectors of a model (i.e. $\alpha,\beta \varepsilon \Omega$), then the coupling of a state in $\alpha$ with one in $\beta$ results in a state $\alpha\beta$ consisting of the Ramond fermions $h^\mu$ which are in either $\alpha$ or $\beta$ - but not both:

$$\alpha\beta = (\alpha \wedge \bar{\beta}) \vee (\bar{\alpha} \wedge \beta) = \alpha \vee \beta - \alpha \wedge \beta \tag{V.6}$$

$$V = \text{union}$$

$$\Lambda = \text{intersection.}$$




The product $\alpha\beta$ is called the symmetric difference of the sets $\alpha, \beta$. It satisfies the rules:

$$\begin{aligned} &(\text{i}) && \alpha\beta = \beta\alpha \\ &(\text{ii}) && (\alpha\beta)\gamma = \alpha(\beta\gamma) \\ &(\text{iii}) && \alpha\emptyset = \alpha \\ &(\text{iv}) && \alpha\alpha = \emptyset. \end{aligned} \tag{V.7}$$

These rules mean that $\Omega$ is an abelian group under this product, with $\emptyset$ as the identity element. Rule (iv) states that every element is of order 2. It then follows that $\Omega \cong (Z_2)^{K+1}$ for some integer $K$, where $Z_2$ denotes the multiplicative group $\{\pm 1\}$, and there are $2^{K+1}$ sectors. $\Omega$ is a subgroup of the group, isomorphic to $(Z_2)^{40}$, formed by all the subsets of $F$. Instead of the set notation one could use 40-component vectors to describe the sectors $\alpha$, with components equal to 1 for NS fields and -1 for R fields. The symmetric difference operation is then given by coordinate-wise multiplication of the vectors. This alternative definition will be used to describe twisted fermions.

Because $\Omega \cong (Z_2)^{K+1}$, the group is generated by a set of elements, $b_0, b_1, \ldots, b_K$,

$$\Omega = \{\emptyset, b_0, b_1, \ldots, b_K;\ b_i b_j,\ (i<j);\ b_i b_j b_k\ (i<j<k);\ \ldots;\ b_0 b_1 \ldots b_K\}. \tag{V.8}$$

It is convenient, when possible, to choose the $b_j$, $0 \leq j \leq K$, so that they do not overlap, i.e. $b_i \wedge b_j = \emptyset$, $i \neq j$. In this case we can describe each sector in the model by K+1 nonoverlapping boxes and draw a diagram for the sector which characterizes it by indicating on the diagram which boxes are R and which are NS and by writing in each box the sign of the number operator corresponding to that box. However, it may not be possible to find a nonoverlapping basis, and in that case we will need more than K+1 boxes, which cannot be taken to be R or NS independently. Furthermore, the sign of the projection operators can then only be specified for combinations of the boxes.

The parity $(-1)^{N_\beta}$ of the number operator $N_\beta$, for the subsets $\beta \in F$ of fermi fields, is defined by

$$\begin{aligned} (-1)^{N_\beta} h_r^\mu &= -h_r^\mu (-1)^{N_\beta}, \quad h_r^\mu \in \beta; \\ (-1)^{N_\beta} h_r^\mu &= h_r^\mu (-1)^{N_\beta}, \quad h_r^\mu \notin \beta; \end{aligned} \tag{V.9}$$

together with a prescription for $N_\beta$ on the vacuum states $|0\rangle_\alpha$ for each sector $\alpha$. If $|\alpha|$ denotes the number of elements in the set $\alpha$, the vacuum states $|0\rangle_\alpha$ provide a representation space, of dimension $2^{\frac{1}{2}|\alpha|}$ (since $|\alpha|$ is even as will

be discussed below), for the Dirac gamma matrices associated with the $|\alpha|$ R fields. The parity operator $(-1)^{N_\beta}$ involves the product of the $|\alpha_\wedge\beta|$ of these gamma matrices associated with the R fields in $\alpha_\wedge\beta$. Two such parity operators $(-1)^{N_\beta}$ and $(-1)^{N_\gamma}$ will commute, when acting on the sector $\alpha$, if both $|\alpha_\wedge\beta_\wedge\gamma|$ and one of $|\alpha_\wedge\beta|$, $|\alpha_\wedge\gamma|$ are even. Unless this condition holds for all $\alpha \in \Omega$ and $\beta,\gamma \in \Omega'$, we will not be able to ascribe values consistently to the various parity operators, and we will not have a consistent set of projections. Thus, we assume this condition holds henceforth, and note that eventually it follows from condition (C2).

If the parities of both $N_\beta$ and $N_\gamma$ are specified on the sector $\alpha$, then it follows that the product of these parities is given by the parity of $N_{\beta_\vee\gamma-\beta_\wedge\gamma}$, which is defined to be $N_{\beta\gamma}$, and so $\Omega'$, as well as $\Omega$, is a group under the symmetric difference operation. To see this we note that the number of elements in $\beta_\vee\gamma$, $|\beta_\vee\gamma| = |\beta| + |\gamma| - |\beta_\wedge\gamma|$, and so we have

$$|\beta\gamma| = |\beta_\vee\gamma| - |\beta_\wedge\gamma| = |\beta| + |\gamma| - 2|\beta_\wedge\gamma|. \qquad \text{(V.10)}$$

It follows that

$$(-1)^{N_{\beta\gamma}}\Big|_\alpha = (-1)^{N_\beta}\Big|_\alpha (-1)^{N_\gamma}\Big|_\alpha, \qquad \text{(V.11a)}$$

i.e.

$$\text{(A)} \qquad \varepsilon(\alpha,\beta\gamma) = \varepsilon(\alpha,\beta)\varepsilon(\alpha,\gamma), \qquad \text{(V.11b)}$$

provided that their definitions agree on the vacuum states $|0\rangle_\alpha$. Later, we will give a definition of $(-1)^{N_\beta}\Big|_\alpha$ and then demonstrate that with this definition eq. (V.11) holds. We shall refer to eq. (V.11) as condition (A). Note that $\varepsilon(\alpha,\emptyset) = 1$.

It is convenient to discuss condition (B), which expresses the conservation of the quantum numbers associated with the various projections at vertices coupling the sectors $\alpha,\beta$ and $\alpha\beta$, later. See eq. (V.35). One general consequence of this conservation condition is that the set of projections made, $\Omega'$, should be the same in each sector.

For consistency, the number operator projections must commute with Lorentz transformations, and since $J^{\hat{i}-} \sim \sum h^{\hat{i}}_{-r} G_r$, this together with eq. (V.9) requires that $G^L$ and $G^R$, the fermionic gauge operators for the left- and right-moving modes, must each have a definite parity, $\delta^L$ and $\delta^R$, respectively, with respect to the operators $(-1)^{N_\beta}$, $\beta \in \Omega'$, i.e.

$$(-1)^{N_\beta} G^L = \delta^L_\beta G^L (-1)^{N_\beta}, \qquad \text{(V.12)}$$

and similarly for $G^R$. Then we set $\delta_\beta = \delta^R_\beta \delta^L_\beta$. For $\alpha \in \Omega$ we define $\delta_\alpha = 1$ if the states of the sector $\alpha$ are spacetime bosons and $\delta_\alpha = -1$ if the states are spacetime fermions. This is the factor that occurs in eq. (V.2) and is just the familiar statistics factor associated with fermion loops.

If $H^\alpha_R \otimes H^{\bar{\alpha}}_{NS}$ denotes the space of states obtained by taking the $h^\mu_r \in \alpha$ to be R and the $h^\mu_r \in \bar{\alpha}$ to be NS, the sector $\alpha$ is obtained by applying the projection operators

$$P_{\alpha,\beta} = \tfrac{1}{2}\{1 + \epsilon(\alpha,\beta)(-1)^{N_\beta}\}, \qquad \beta \in \Omega' \tag{V.13}$$

to these states. Since $(-1)^{N_{\beta\gamma}} = (-1)^{N_\beta}(-1)^{N_\gamma}$, it follows that

$$P_{\alpha,b_{i_1}b_{i_2}\ldots b_{i_m}} P_{\alpha,b_{i_1}} P_{\alpha,b_{i_2}} \ldots P_{\alpha,b_{i_m}} = P_{\alpha,b_{i_1}} P_{\alpha,b_{i_2}} \ldots P_{\alpha,b_{i_m}}, \tag{V.14}$$

where here $b_o, b_1, \ldots, b_{K'}$ is a basis of generators for $\Omega'$, as we had in eq. (V.8) for $\Omega$. From this we may deduce

$$\begin{aligned}
\prod_{\beta\in\Omega'} P_{\alpha,\beta} &= \prod_{i=0}^{K'} P_{\alpha,b_i} \\
&= \frac{1}{2^{K'+1}}\{1 + \sum_j \epsilon(\alpha,b_j)(-1)^{N_{b_j}} + \sum_{j<k} \epsilon(\alpha,b_j)\epsilon(\alpha,b_k)(-1)^{N_{b_j} + N_{b_k}} + \ldots\} \\
&= \frac{1}{2^{K'+1}}\{1 + \sum_j \epsilon(\alpha,b_j)(-1)^{N_{b_j}} + \sum_{j<k} \epsilon(\alpha,b_jb_k)(-1)^{N_{b_jb_k}} + \ldots\} \\
&= \frac{1}{2^{K'+1}} \sum_{\beta\in\Omega'} \epsilon(\alpha,\beta)(-1)^{N_\beta}.
\end{aligned} \tag{V.15}$$

Eq. (V.2) can now be expressed as

$$\Lambda_f = \frac{1}{2^{K'+1}} \sum_{\alpha\in\Omega} \sum_{\beta\in\Omega'} \delta_\alpha \epsilon(\alpha,\beta)\, \mathrm{tr}_\alpha\{\bar{w}^{\bar{L}_o - \frac{1}{2}} w^{L_o - \frac{1}{2}} (-1)^{N_\beta}\}. \tag{V.16}$$

This is usually described as a sum over spin structures and in deducing this form from eq. (V.2) we have reversed an argument in ref. [35]. Denote the trace in eq. (V.16) by $\{\alpha,\beta\}$, so that, ignoring the factor of $2^{-K'-1}$, the sum becomes

$$\sum_{\alpha,\beta} \delta_\alpha \varepsilon(\alpha,\beta)\{\alpha,\beta\}, \tag{V.17}$$

where

$$\begin{aligned}\{\alpha,\beta\} &= \mathrm{tr}_\alpha\{\bar{w}^{\bar{L}_0-\frac{1}{2}} w^{L_0-\frac{1}{2}} (-1)^{N_\beta}\} \\ &= |w|^{-1}|f(w)|^{-20} \prod_{i=1}^{20} [\bar{\Theta}\binom{\tilde{\lambda}_\alpha}{\tilde{\lambda}_\beta}(0|\tau)]^{1/2} \prod_{i=1}^{20} [\Theta\binom{\lambda_\alpha}{\lambda_\beta}(0|\tau)]^{1/2}. \end{aligned} \tag{V.18}$$

$\lambda_\alpha$ and $\lambda_\beta$ are the exponents in the forty-component vectors describing the sectors $\alpha$ and $\beta$ respectively, i.e. the components are 0 for NS and 1 for R, and the boundary conditions are given by $h^\mu \to -e^{i\pi\lambda_\alpha} h^\mu$. $\{\alpha,\beta\}$ must transform under modular transformations in such a way that the integrand of eq. (V.1) is invariant.

Since[42]

$$\Theta\binom{\lambda}{\mu}(0|\tau+1) = e^{i\frac{\pi}{4}\lambda^2}\,\Theta\binom{\lambda}{\lambda+\mu-1}(0|\tau) \tag{V.19a}$$

and

$$\Theta\binom{\lambda}{\mu}(0|\tfrac{-1}{\tau}) = (-i\tau)^{1/2} e^{-i\frac{\pi}{2}\lambda\mu}\,\Theta\binom{\mu}{-\lambda}(0|\tau) \tag{V.19b}$$

it is straightforward to show that under modular transformations:

$$\tau\to\tau+1: \qquad \{\alpha,\beta\} \to e^{-i\frac{\pi}{8}n(\alpha)}\{\alpha,\bar{\alpha}\beta\} \tag{V.20a}$$

$$\tau\to\frac{-1}{\tau}: \qquad \{\alpha,\beta\} \to e^{i\frac{\pi}{4}n(\alpha_\Lambda\beta)}\{\beta,\alpha\} \tag{V.20b}$$

where $n(\alpha) = n^L(\alpha) - n^R(\alpha)$ and $n^L(\alpha)$, $n^R(\alpha)$ denote the number of left and right moving Ramond fermions in $\alpha$, respectively. The transformation (V.20b) actually has additional factors which cancel with factors coming from the rest of the partition function. So the partition function is invariant under the modular transformation $\tau\to\tau+1$ if

$$\begin{aligned}\sum_{\alpha,\beta} \delta_\alpha \epsilon(\alpha,\beta)\{\alpha,\beta\} &= \sum_{\alpha,\beta} \delta_\alpha \epsilon(\alpha,\beta)\Sigma_\alpha\{\alpha,\bar{\alpha}\beta\} \\ &= \sum_{\alpha,\beta} \delta_\alpha \epsilon(\alpha,\bar{\alpha}\beta)\Sigma_\alpha\{\alpha,\beta\},\end{aligned} \tag{V.21a}$$

where

$$\Sigma_\alpha = e^{-i\frac{\pi}{8}n(\alpha)}. \tag{V.21b}$$

This requires that

$$\text{(C1)} \qquad \epsilon(\alpha,\beta) = \Sigma_\alpha \epsilon(\alpha,\bar{\alpha}\beta). \tag{V.22}$$

The second modular transformation $\tau\to-1/\tau$ causes the partition function to transform as

$$\sum_{\alpha,\beta} \delta_\alpha \varepsilon(\alpha,\beta)\{\alpha,\beta\} = \sum_{\alpha,\beta} \delta_\alpha \varepsilon(\alpha,\beta)\Sigma^{-2}_{\alpha_\Lambda\beta}\{\beta,\alpha\}$$
$$= \sum_{\alpha,\beta} \delta_\beta \varepsilon(\beta,\alpha)\Sigma^{-2}_{\alpha_\Lambda\beta}\{\alpha,\beta\}, \tag{V.23}$$

which is invariant if the following condition holds

$$\text{(C2)} \qquad \varepsilon(\alpha,\beta) = \delta_\alpha\delta_\beta\varepsilon(\beta,\alpha)\Sigma^{-2}_{\alpha_\Lambda\beta}. \tag{V.24}$$

The consistency of (V.22) and (V.24) requires that both $\Sigma_\alpha$ and $\Sigma^2_{\alpha_\Lambda\beta}$ are each either $\pm 1$, and this implies that $n(\alpha) \in 8Z$ and $n(\alpha_\Lambda\beta) \in 4Z$, for all $\alpha,\beta \in \Omega$. We shall refer to (V.22) as condition (C1) and (V.24) as condition (C2). To satisfy condition (C1) we clearly need that $\Omega = \Omega'$, so we drop the distinction henceforth. Conditions (C1) and (C2) are what is required for the modular invariance of N-point one-loop integrands. Together with condition (A), they are equivalent to the conditions Antoniadis, Bachas, and Kounnas[35] obtained for multi-loop modular invariance.

Condition (C1) shows that $\bar\alpha = \alpha F \in \Omega$ and so necessarily $F \in \Omega$ for modular invariance. Since $\varepsilon(\alpha,\emptyset) = 1$, from eq. (V.24) we get $\varepsilon(\emptyset,\alpha) = \delta_\alpha$. This implies that the graviton is always in the spectrum[35], since this is composed of states in $\emptyset$, namely $b^{\hat{i}}_{L,-\frac{1}{2}} b^{\hat{j}}_{R,-\frac{1}{2}}|0\rangle$, on which $(-1)^{N_\alpha}$ will take the value $\delta_\alpha$, for any sector $\alpha$, and so they survive the projection.

In writing a general solution to conditions (A), (C1), and (C2), it is convenient to use $b_o = F$ as one of the basis elements. Then we may express all the $\varepsilon(\alpha,\beta)$ in terms of the quantities

$$\eta_{ij} = \varepsilon(b_j, b_i)\delta_{b_i} \tag{V.25}$$

for $0 \leqq i < j \leqq K$ and $i = j = 0$. If $\alpha = \prod_{i\varepsilon I} b_i$ and $\beta = \prod_{j\varepsilon J} b_j$, the solution is

$$\varepsilon(\alpha,\beta) = \delta_\beta \prod_{\substack{i\varepsilon I \\ j\varepsilon J}} \eta_{ij} \Sigma^2_{\alpha_\Lambda b_j}. \tag{V.26}$$

Since

$$\Sigma_{\alpha_\Lambda \beta\gamma} = \Sigma_{\alpha_\Lambda\beta} \Sigma_{\alpha_\Lambda\gamma} \Sigma^{-2}_{\alpha_\Lambda\beta_\Lambda\gamma}, \tag{V.27}$$

and $n(\alpha_\Lambda\beta) \ \varepsilon \ 4Z$, it follows that $n(\alpha_\Lambda\beta_\Lambda\gamma) \ \varepsilon \ 2Z$, for any $\alpha,\beta,\gamma \ \varepsilon \ \Omega$. For eq. (V.26) to satisfy conditions (A), (C1), and (C2), we need the consistency condition $n(\alpha_\Lambda\beta_\Lambda\gamma_\Lambda\delta) \ \varepsilon \ 2Z$ as well as $n(\alpha_\Lambda\beta) \ \varepsilon \ 4Z$, $n(\alpha) \ \varepsilon \ 8Z$ for all $\alpha,\beta,\gamma,\delta \ \varepsilon \ \Omega$. If it should happen that $n(\alpha_\Lambda\beta_\Lambda\gamma) \ \varepsilon \ 4Z$ then eq. (V.26) can be replaced by the simpler formula

$$\varepsilon(\alpha,\beta) = \delta_\beta \prod_{\substack{i\varepsilon I \\ j\varepsilon J}} \tilde{\eta}_{ij}, \tag{V.28}$$

where

$$\tilde{\eta}_{ij} = \eta_{ij} \Sigma^2_{b_{i\Lambda} b_j} = \varepsilon(b_j, b_i) \Sigma^2_{b_{i\Lambda} b_j} \delta_{b_i}. \tag{V.29}$$

From eq. (V.24) it follows that $\eta_{ij} = \Sigma^2_{b_i \wedge b_j} \eta_{ji}$, and from eq. (V.22) that $\eta_{io} = \delta_{b_i} \Sigma_{b_i} \eta_{ii}$. If the $\eta$'s in eq. (V.26) have these properties, it follows[35] that $\varepsilon(\alpha,\beta)$ satisfies (A), (C1), and (C2) provided that $n(\alpha_\wedge \beta_\wedge \gamma_\wedge \delta) \in 2Z$ for all $\alpha,\beta,\gamma,\delta \in \Omega$.

Applying the interchange property eq. (V.24) to eq. (V.11), and using $\Sigma_{\alpha_\wedge \beta\gamma} = \Sigma_{\alpha_\wedge \beta} \Sigma_{\alpha_\wedge \gamma} \Sigma^{-2}_{\alpha_\wedge \beta_\wedge \gamma}$, we obtain

$$\varepsilon(\beta\gamma,\alpha) = \varepsilon(\beta,\alpha)\varepsilon(\gamma,\alpha)\Sigma^4_{\alpha_\wedge \beta_\wedge \gamma}\delta_\alpha. \tag{V.30}$$

Since this equation relates the values of the parity of $N_\alpha$ in the various sectors that meet at the vertex $\beta + \gamma \to \beta\gamma$, it must express the conservation of that quantum number at that vertex or it is inconsistent. We shall show there is no such inconsistency when both $n^L(\alpha_\wedge \beta_\wedge \gamma)$ and $n^R(\alpha_\wedge \beta_\wedge \gamma)$ are even and not just $n(\alpha_\wedge \beta_\wedge \gamma)$. Since we have already shown that $n(\alpha_\wedge \beta_\wedge \gamma) = 2Z$, and since $n = n^L - n^R$, it must be that $n^L(\alpha_\wedge \beta_\wedge \gamma)$ and $n^R(\alpha_\wedge \beta_\wedge \gamma)$ are either both even or both odd. For the case where $n^L(\alpha_\wedge \beta_\wedge \gamma)$ and $n^R(\alpha_\wedge \beta_\wedge \gamma)$ are both even, we can consistently define the parity operators and can prove condition (A). However, if $n^L(\alpha_\wedge \beta_\wedge \gamma)$ and $n^R(\alpha_\wedge \beta_\wedge \gamma)$ are odd, then we do not yet know how to consistently define the parity of the number operators. This last case is still under investigation.

One might naively have expected just $\varepsilon(\beta\gamma,\alpha) = \varepsilon(\beta,\alpha)\varepsilon(\gamma,\alpha)$ to express the conservation law at the vertex, but the factors of $\delta_\alpha$ and $\Sigma^4_{\alpha_\wedge \beta_\wedge \gamma}$ come in for the reasons to be explained now.

The conservation law corresponding to the vertex $\beta + \gamma \to \beta\gamma$ is most easily determined for the case when we can bosonize the 40 fermions in pairs, choosing these pairs so that the fermions in them consist of either both NS or both R fermions in each of the sectors $\beta$, $\gamma$, $\beta\gamma$. Then we obtain an (untwisted) boson for each fermion pair, with momentum quantized on Z in the NS case and $Z+\frac{1}{2}$ in the R case. We denote the resulting 20 component momentum vector by $p^\alpha$ in the sector $\alpha$. In these bosonized terms, we can give a definition of the parity of the number operator $N_\beta$, acting on the sector $\alpha$, as

$$(-1)^{N_\beta}\Big|_\alpha = (-1)^{\sum\limits_{j\varepsilon\beta} p_j^\alpha - \frac{1}{4}|\alpha_\wedge\beta|} . \qquad \text{(V.31)}$$

In this equation, we are using $\alpha$ to denote the set of the $\frac{1}{2}|\alpha|$ components of the momentum vectors corresponding to the fermions in $\alpha$. The final term in the exponent corresponds to a convention for assigning the value of the parity of $N_\beta$ on the fermion vacuum. From the definition of the number operator in (V.31), we can prove condition (A) as follows.

We need only verify that $N_\beta + N_\gamma - N_{\beta\gamma}$ acting on any sector is always even. In the sector $\alpha$, this difference is

$$\begin{aligned} N_\beta + N_\gamma - N_{\beta\gamma} &= \sum_{j\varepsilon\beta} p_j^\alpha + \sum_{j\varepsilon\gamma} p_j^\alpha - \sum_{j\varepsilon\beta\gamma} p_j^\alpha - \tfrac{1}{4}|\alpha_\wedge\beta| - \tfrac{1}{4}|\alpha_\wedge\gamma| + \tfrac{1}{4}|\alpha_\wedge\beta\gamma| \\ &= 2\sum_{j\varepsilon\beta_\wedge\gamma} p_j^\alpha - \tfrac{1}{2}|\alpha_\wedge\beta_\wedge\gamma| . \end{aligned} \qquad \text{(V.32)}$$

Since the second term, $\frac{1}{2}|\alpha_\wedge\beta_\wedge\gamma|$ ,is the number of components of $p^\alpha$ in the sum corresponding to R fields, with each such value contributing a $\frac{1}{2}$, this term

will exactly cancel the odd part of the first term, and the difference will be even, i.e.

$$N_\beta\Big|_\alpha + N_\gamma\Big|_\alpha = N_{\beta\gamma}\Big|_\alpha \qquad (\text{mod } 2). \qquad (\text{V.33})$$

To establish (V.30) we must consider the number operator $N_\alpha$ acting on different sectors:

$$(\text{B}) \qquad N_\alpha\Big|_\beta + N_\alpha\Big|_\gamma - N_\alpha\Big|_{\beta\gamma} = \tfrac{-1}{2}|\alpha_\wedge\beta_\wedge\gamma|. \qquad (\text{V.34})$$

Here we use the momentum conservation, $p^\beta + p^\gamma = p^{\beta\gamma}$. Now, in eq. (V.30), $\Sigma^4_{\alpha_\wedge\beta_\wedge\gamma}$ equals $\pm 1$ depending on the parity of $\frac{1}{2}n(\alpha_\wedge\beta_\wedge\gamma)$; hence eqs. (V.30) and (V.34) will be equivalent if and only if $n^L(\alpha_\wedge\beta_\wedge\gamma)$ and $n^R(\alpha_\wedge\beta_\wedge\gamma)$ are both even, not just $n(\alpha_\wedge\beta_\wedge\gamma)$. This is the assumption we have already made, and in the case where we can bosonize the fermions this extra condition is automatically true.

In the case where we cannot permanently pair and bosonize the fermions, we must consistently define the number operators in terms of the fermions themselves. Here we introduce another definition that again leads to condition (A).

In the fermionic picture, we define the parity of the number operator $N_\beta$ acting on the sector $\alpha$ by

$$(-1)^{N_\beta}\Big|_\alpha = (2^{\frac{s}{2}})(i)(-1)^{N_b + N_d} \prod_{j=1}^{s} d_o^j, \qquad (\text{V.35a})$$

where

$$N_b = \sum_{j=1}^{r} \sum_{s=1/2}^{\infty} b^j_{-s} b^j_s , \qquad N_d = \sum_{j=1}^{s} \sum_{n=1}^{\infty} d^j_{-n} d^j_n . \tag{V.35b}$$

The NS fields $b^j$, $1 \leq j \leq r$, are those in $\bar{\alpha}_\Lambda \beta$, and the R fields $d^j$, $1 \leq j \leq s$, are those in $\alpha_\Lambda \beta$. The product $\prod_{j=1}^{s} d^j_o$ is defined to be taken in the particular order $d^1_o d^2_o \ldots d^s_0$ (this fixes a sign convention), and the (i) is a factor which is inserted only if $\prod_{j=1}^{s} d^j_o$ is anti-hermitian. With this definition, it is straightforward to show that

$$(-1)^{N_{\beta_1}}\Big|_\alpha (-1)^{N_{\beta_2}}\Big|_\alpha = \pm (-1)^{N_{\beta_1 \beta_2}}\Big|_\alpha . \tag{V.36a}$$

We now eliminate the minus sign in (V.36a) in the following way. We use (V.35a) to define $N_\beta$ for a set of generators of $\Omega$. For the other elements of $\Omega$, we adjust the overall sign of (V.35a) by setting

$$(-1)^{N_{b_{i_1} b_{i_2} \ldots b_{i_n}}}\Big|_\alpha = (-1)^{N_{b_{i_1}}}\Big|_\alpha (-1)^{N_{b_{i_2}}}\Big|_\alpha \cdots (-1)^{N_{b_{i_n}}}\Big|_\alpha . \tag{V.36b}$$

Then eq. (V.36a) is replaced by

$$(-1)^{N_{\beta_1}}\Big|_\alpha (-1)^{N_{\beta_2}}\Big|_\alpha = (-1)^{N_{\beta_1 \beta_2}}\Big|_\alpha . \tag{V.36c}$$

This again leads to condition (A).

In both the fermionic and bosonized cases with $n^L(\alpha_\wedge\beta_\wedge\gamma)$ and $n^R(\alpha_\wedge\beta_\wedge\gamma)$ both even, we may unambiguously define the parity operators. In the remaining cases, with $n^L$ and $n^R$ odd, we do not yet have a well-defined expression for the parity of the number operator.

To summarize this section on untwisted fermions, a string model is thus defined for a given nonnegative integer K by a choice of K+1 generator sectors, together with the values ±1 of $\frac{1}{2}K(K+1)+1$ projections $\eta_{ij}$, for i=j=0 and $0\leqq j<i\leqq K$, which may be chosen independently as long as $n(b_i) \in 8Z$, $n(b_{i\wedge}b_j) \in 4Z$, and $n(b_{i\wedge}b_{j\wedge}b_{k\wedge}b_l) \in 2Z$. The remaining $\eta_{ij}$ are given in terms of $\eta_{oo}$ and $\eta_{ij}$, i>j, by

$$\eta_{ij} = \Sigma^2_{b_{i\wedge}b_j}\eta_{ji}, \tag{V.37a}$$

so that $\eta_{oi} = \eta_{io}$, and

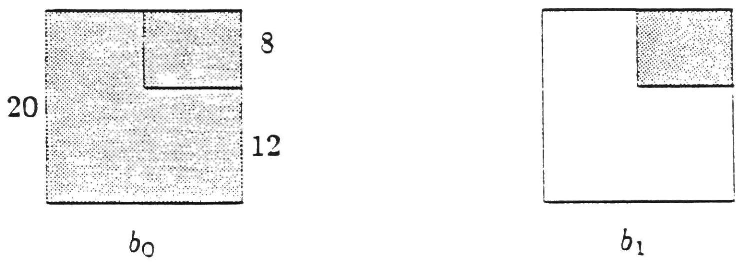
$$\eta_{ii} = \delta_{b_i}\Sigma_{b_i}\eta_{io}. \tag{V.37b}$$

The $2^{K+1}$ sectors are given by $\Omega = \{\emptyset;\ b_o,b_1,\ldots,b_K;\ b_ib_j\ (i<j);\ldots;b_ob_1\ldots b_K\}$. The values of the projections on the various sectors are given in terms of the $\eta_{ij}$ by eq. (V.26) and the contribution to the partition function is given in terms of the $\varepsilon(\alpha,\beta)$ and $\Omega$ by eq. (V.16).

We now give some examples of type II models in four dimensions using untwisted fermions.

EXAMPLE 1

The original N=4 model, with any of the gauge groups $SU(2)^6$, $SU(2)\times SU(4)$, and $SU(3)\times SO(5)$, is described by K=1. The two generator sectors are given by $b_o = F$ and $b_1 = (1^{20};(-1)^8,1^{12})$, where the exponents indicate that the entry occurs that number of times. These sectors can be represented pictorially by

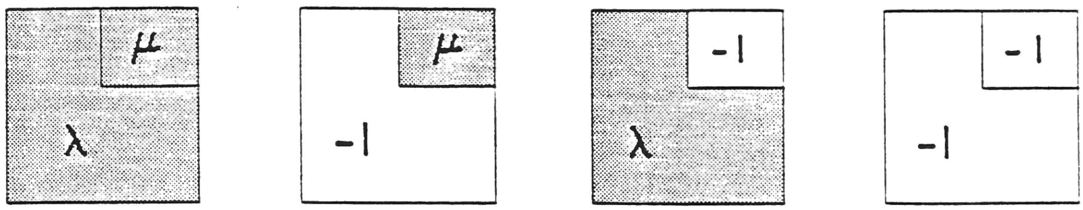



where the shaded boxes represent R fermions and the unshaded boxes represent NS fermions, each box containing fermions in a particular projection. The top two fermions on the left and right sides of each box diagram correspond to the spacetime components $\tilde{h}^{\hat{1}}_r$, $h^{\hat{1}}_r$, respectively. There are two free parameters: $\eta_{oo} = \mu\lambda$, and $\eta_{1o} = -\mu$, where $\lambda,\mu = \pm 1$. The remaining $\eta_{ij} = \varepsilon(b_j,b_i)\delta_{b_i}$ are calculated from eq. (V.37a,b) and are given in the following table.

$\beta \longrightarrow$

| $\alpha \downarrow$ $\epsilon(\alpha,\beta)$ | $\emptyset$ | $b_0$ | $b_1$ | $b_0b_1$ |
|---|---|---|---|---|
| $\emptyset$ | 1 | 1 | $-1$ | $-1$ |
| $b_0$ | 1 | $\lambda\mu$ | $\mu$ | $\lambda$ |
| $b_1$ | 1 | $-\mu$ | $\mu$ | $-1$ |
| $b_0b_1$ | 1 | $-\lambda$ | $-1$ | $\lambda$ |

We can represent the above model by including the projections in the box diagrams.

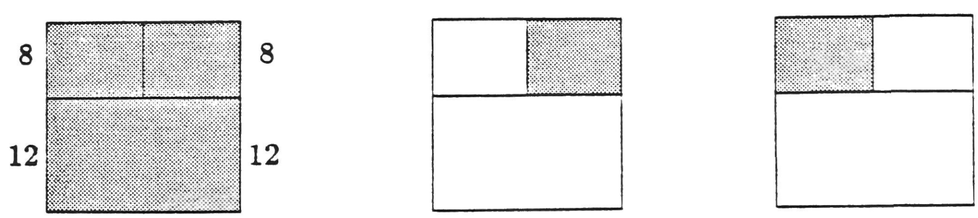



$b_0$ $\qquad$ $b_1$ $\qquad$ $b_0 b_1$ $\qquad$ $\emptyset$

This model consists of four sectors $L = \{\emptyset, b_0, b_1, b_0 b_1\}$. Different choices of $\lambda, \mu$ do not change the model, and $\{b_1, b_0 b_1\}$ constitutes a nonoverlapping basis. The massless states are easily read off from the figure above. They come from the sectors $b_1$ and $\emptyset$, and form an N=4 supergravity multiplet coupled to an N=4 supersymmetric Yang-Mills multiplet in the adjoint representation of a semi-simple gauge group G, with dim G = 18. The spin content is $\{\pm 2, 4(\pm\frac{3}{2}), 6(\pm 1), 4(\pm\frac{1}{2}), 2(0)\}$ and $\{\pm 1, 4(\pm\frac{1}{2}), 6(0)\}$ in the adjoint representation of G.

EXAMPLE 2

An N=8 model is described by K=2, and the following generators.



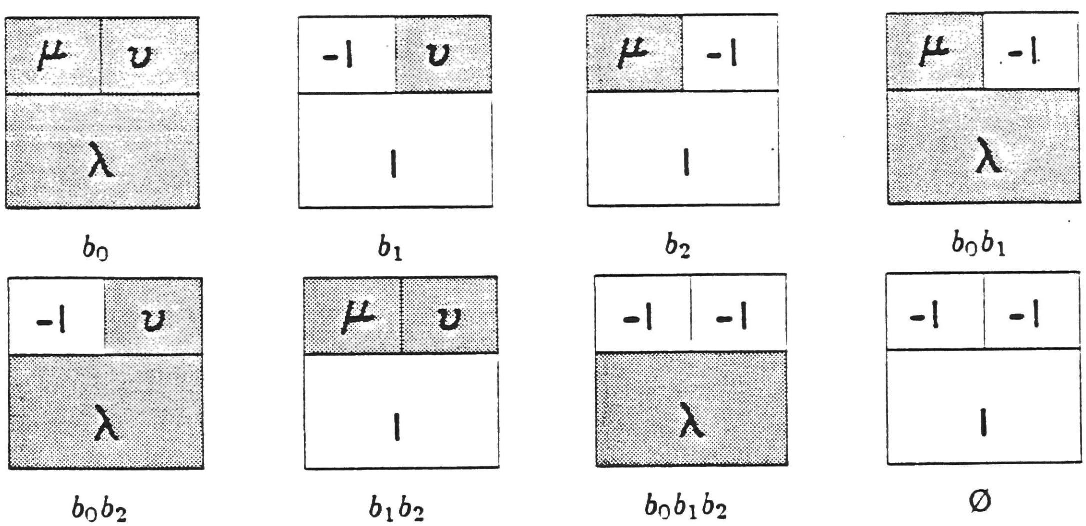

$b_0$ $\qquad$ $b_1$ $\qquad$ $b_2$

There are four free parameters. We choose $\eta_{00} = \mu\nu\lambda$, $\eta_{01} = -\nu$, $\eta_{21} = 1$, and $\eta_{20} = -\mu$. Different choices of the $\lambda,\mu,\nu$ do not change the model. These projections are given in the table.

$\beta \longrightarrow$ ; $\alpha \downarrow$

| $\epsilon(\alpha,\beta)$ | $\emptyset$ | $b_0$ | $b_1$ | $b_2$ | $b_0b_1$ | $b_0b_2$ | $b_1b_2$ | $b_0b_1b_2$ |
|---|---|---|---|---|---|---|---|---|
| $\emptyset$ | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ | 1 | 1 |
| $b_0$ | 1 | $\lambda\mu\nu$ | $\nu$ | $\mu$ | $\lambda\mu$ | $\lambda\nu$ | $\mu\nu$ | $\lambda$ |
| $b_1$ | 1 | $-\nu$ | $\nu$ | $-1$ | $-1$ | $\nu$ | $-\nu$ | 1 |
| $b_2$ | 1 | $-\mu$ | $-1$ | $\mu$ | $\mu$ | $-1$ | $-\mu$ | 1 |
| $b_0b_1$ | 1 | $-\lambda\mu$ | $-1$ | $\mu$ | $\lambda\mu$ | $-\lambda$ | $-\mu$ | $\lambda$ |
| $b_0b_2$ | 1 | $-\lambda\nu$ | $\nu$ | $-1$ | $-\lambda$ | $\lambda\nu$ | $-\nu$ | $\lambda$ |
| $b_1b_2$ | 1 | $\mu\nu$ | $\nu$ | $\mu$ | $\mu$ | $\nu$ | $\mu\nu$ | 1 |
| $b_0b_1b_2$ | 1 | $\lambda$ | $-1$ | $-1$ | $-\lambda$ | $-\lambda$ | 1 | $\lambda$ |

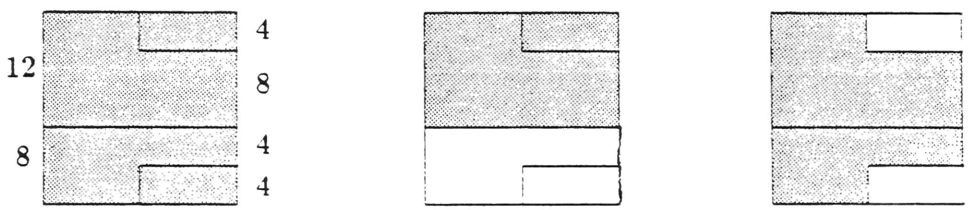
This can be represented by the box diagrams.




The massless states come from the sectors $b_1, b_2, b_1b_2, \emptyset$, and form a four-dimensional N=8 supergravity multiplet $\{\pm 2, 8(\pm\frac{3}{2}), 28(\pm 1), 8(\pm\frac{1}{2}), 70(0)\}$.

EXAMPLE 3

An N=2, SU(3)×U(1)×U(1) model with K=2 can be constructed as follows. There are 8 sectors, generated from the three generators $b_o$, $b_1$, and $b_2$.




$b_0$ $b_1$ $b_2$

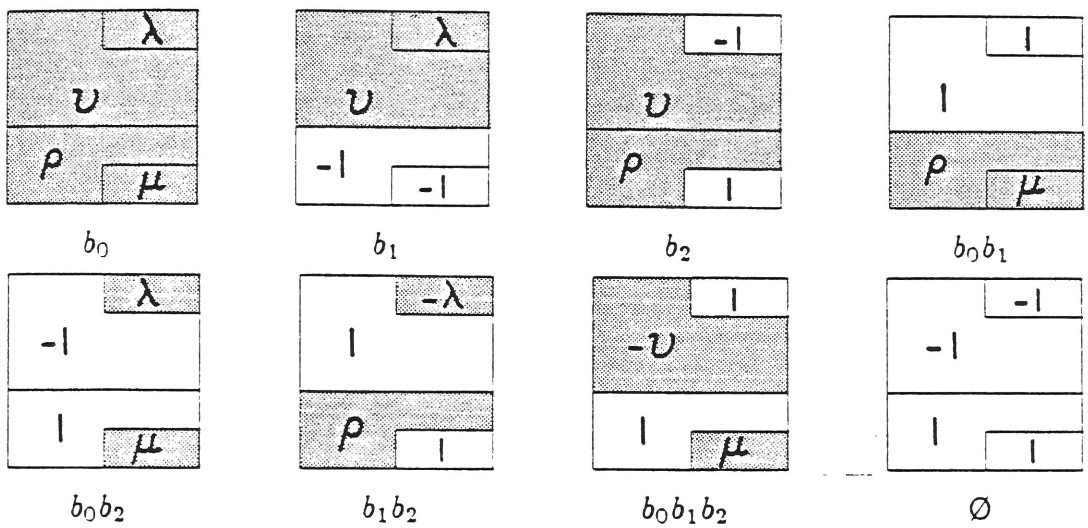
In this case it is not possible to choose a nonoverlapping basis - hence the need for four subboxes per sector (which will always be projected on in pairs). We label the four free parameters $\eta_{1o} = -\lambda\mu$, $\eta_{2o} = \lambda\nu$, $\eta_{21} = -\lambda$, and $\eta_{oo} = \lambda\mu\nu\varrho$. The projections are then calculated as shown in the table.

$\beta \longrightarrow$

$\alpha \downarrow$

| $\epsilon(\alpha,\beta)$ | $\emptyset$ | $b_0$ | $b_1$ | $b_2$ | $b_0b_1$ | $b_0b_2$ | $b_1b_2$ | $b_0b_1b_2$ |
|---|---|---|---|---|---|---|---|---|
| $\emptyset$ | 1 | 1 | 1 | $-1$ | 1 | $-1$ | $-1$ | $-1$ |
| $b_0$ | 1 | $\lambda\mu\nu\rho$ | $\lambda\nu$ | $\nu\rho$ | $\mu\rho$ | $\lambda\mu$ | $\lambda\rho$ | $\mu\nu$ |
| $b_1$ | 1 | $\lambda\nu$ | $\lambda\nu$ | $-\nu$ | 1 | $-\lambda$ | $-\lambda$ | $-\nu$ |
| $b_2$ | 1 | $-\nu\rho$ | $-\nu$ | $\nu\rho$ | $\rho$ | $-1$ | $-\rho$ | $\nu$ |
| $b_0b_1$ | 1 | $\mu\rho$ | 1 | $\rho$ | $\mu\rho$ | $\mu$ | $\rho$ | $\mu$ |
| $b_0b_2$ | 1 | $-\lambda\mu$ | $-\lambda$ | $-1$ | $\mu$ | $\lambda\mu$ | $\lambda$ | $-\mu$ |
| $b_1b_2$ | 1 | $-\lambda\rho$ | $-\lambda$ | $\rho$ | $\rho$ | $-\lambda$ | $-\lambda\rho$ | 1 |
| $b_0b_1b_2$ | 1 | $-\mu\nu$ | $-\nu$ | $-\nu$ | $\mu$ | $\mu$ | 1 | $-\mu\nu$ |

We can again incorporate the table into the box diagrams. Here $\epsilon(\alpha,\beta)$ is given by the product of the signs in the boxes corresponding to $\beta$ in the diagram labelled by $\alpha$ (i.e. describing the sector $\alpha$).




In this model, each box diagram in the figure contributes twice to the partition function, once as labelled, and once with a minus sign on all the labels.

The massless states come from the sectors $b_0b_1, b_0b_2, b_1b_2, \emptyset$. They form an N=2 supergravity multiplet with spins $\{\pm2, 2(\pm\frac{3}{2}), \pm1\}$, and an N=2 super-Yang-Mills multiplet $\{\pm1, 2(\pm\frac{1}{2}), 2(0)\}$ in a singlet and adjoint representation of SU(3)xU(1)xU(1), and an N=2 spin-1/2 multiplet of spin $\{2(\pm\frac{1}{2}), 2(0)\}$ in the

$$(\bar{3},1,0)\oplus(\bar{3},-1,0)\oplus(1,0,\pm1)\oplus2[(3,\tfrac{1}{3},\pm\tfrac{1}{2})\oplus(1,-1,\pm\tfrac{1}{2})\oplus(\bar{3},\tfrac{-1}{3},\pm\tfrac{1}{2})\oplus(1,1,\pm\tfrac{1}{2})] \qquad (V.38)$$

of SU(3)xU(1)xU(1). These last fermion representations are similar to the $(3,\frac{1}{3},2)\oplus(1,-1,2)$ quark and neutrino doublets of the standard model SU(3)xU(1)xSU(2).

Twisted Fermions

The untwisted fermions described in the previous section correspond to bosons compactified with internal momenta $\sqrt{2\alpha'}p \in Z$ for the NS case and $\sqrt{2\alpha'}p \in Z + \frac{1}{2}$ for the R case. Models with chiral spacetime fermions can be described if we allow some of the internal coordinates to take values on a shifted lattice $\sqrt{2\alpha'}p \in Z + \frac{\lambda_\alpha}{2}$, which now corresponds to a complex twisted fermion satisfying the boundary condition:

$$f(e^{2\pi i}z) = -\alpha(f)f(z) = -e^{i\pi\lambda_\alpha}f(z) \tag{V.39a}$$

and consequently

$$f^*(e^{2\pi i}z) = -\alpha^*(f)f^*(z) = -e^{-i\pi\lambda_\alpha}f^*(z). \tag{V.39b}$$

The corresponding oscillators are fractionally moded,

$$f(z) = \sum_{r\in Z+\Delta} f_r z^{-r} \qquad\qquad f^*(z) = \sum_{r\in Z-\Delta} \tilde{f}_r z^{-r} \tag{V.40}$$

where $\Delta = \frac{1-\lambda_\alpha}{2}$ and $f_r^\dagger = \tilde{f}_{-r}$. It is sufficient to consider $-1 < \lambda_\alpha \leq 1$; $\lambda=1$ is the Ramond case and $\lambda=0$ is the NS case. The oscillators have anti-commutation relations

$$\{f^i_r, \tilde{f}^j_s\} = \delta^{ij}\delta_{r,-s}; \qquad \{f^i_r, f^j_s\} = \{\tilde{f}^i_r, \tilde{f}^j_s\} = 0. \tag{V.41}$$

The number of states in the Fock space generated by $f_r$ and $\tilde{f}_s$ is given by the generalized Jacobi theta function

$$\begin{aligned}\prod_{r=\Delta}^{\infty}(1+w^r)\prod_{s=1-\Delta}^{\infty}(1+w^s) &= w^{-\frac{1}{8}\lambda_\alpha^2}[f(w)]^{-1}\sum_{n\varepsilon Z} w^{\frac{1}{2}(n+\frac{\lambda_a}{2})^2} \\ &= w^{-\frac{1}{8}\lambda_\alpha^2}[f(w)]^{-1}\,\Theta\left(\begin{smallmatrix}\lambda_\alpha\\ 0\end{smallmatrix}\right)(0|\tau)\end{aligned} \tag{V.42}$$

where

$$\Theta\left(\begin{smallmatrix}\zeta\\ \mu\end{smallmatrix}\right)(\nu|\tau) = \sum_{n\varepsilon Z} e^{i\pi\tau(n+\frac{\zeta}{2})^2}\, e^{i2\pi(n+\frac{\zeta}{2})(\nu+\frac{\mu}{2}) - i\frac{\pi}{2}\zeta\mu} \tag{V.43a}$$

$$\Theta\left(\begin{smallmatrix}\tilde{\zeta}\\ \tilde{\mu}\end{smallmatrix}\right)(\nu|\tau) = \sum_{n\varepsilon Z} e^{-i\pi\tilde{\tau}(n+\frac{\tilde{\zeta}}{2})^2}\, e^{-i2\pi(n+\frac{\tilde{\zeta}}{2})(\nu+\frac{\tilde{\mu}}{2}) + i\frac{\pi}{2}\tilde{\zeta}\tilde{\mu}} \tag{V.43b}$$

$$f(w) = \prod_{n=1}^{\infty}(1-w^n). \tag{V.43c}$$

The Hamiltonian and associated Virasoro generators are given by

$$L_n = \sum_{n\varepsilon Z+\Delta}(r - \frac{n}{2}):\tilde{f}_{n-r}f_r: + \frac{1}{4}\sum(\lambda - \frac{1}{2})^2\delta_{n.o} \tag{V.44}$$

where the second term is a sum over all $f$ and $\tilde{f}$ seperately.

A string model with some of the internal modes given by twisted fermions can again be described by eq. (V.2) where a sector $\alpha$ is now labelled by the values of $\lambda_\alpha(f)$ - the exponent in the boundary condition in eq. (V.39) - for each of the fermions, and the definition of $P_{\alpha,\beta}$ is modified. In general the left- and right-moving modes may seperately be described by m complex fermions, each of whose boundary condition (V.39) may be labelled by one real value of $\lambda_\alpha(f)$; and n real fermions whose boundary conditions need not be paired, with $2m + n = 20$. A sector $\alpha$ is thus labelled by a $(m+n+m'+n')$-dimensional vector

$$\vec{\lambda}_\alpha = (\tilde{\lambda}_1,\ldots,\tilde{\lambda}_n;\tilde{\lambda}'_1,\ldots,\tilde{\lambda}'_m;\lambda_1,\ldots,\lambda_{n'};\lambda'_1,\ldots,\lambda'_{m'}). \qquad (V.45)$$

We consider boundary conditions in eq. (V.39) of finite order, i.e.

$$\alpha(f)^N = e^{i\pi\lambda_\alpha(f)N} = 1$$

for some integer N. The vector of exponentials of eq. (V.45), which reduces to the 40-component vector, with components equal to $\pm 1$ in the untwisted case, will be denoted by

$$\alpha = (e^{i\pi\tilde{\lambda}_1},\ldots;e^{i\pi\tilde{\lambda}'_1},\ldots;e^{i\pi\lambda_1},\ldots;e^{i\pi\lambda'_1},\ldots). \qquad (V.46)$$

The entries in such vectors are complex numbers of unit modulus, so we can define a multiplicative operation on such vectors by multiplying the components,

$$(\alpha_1,\alpha_2,\ldots)(\beta_1,\beta_2,\ldots) = (\alpha_1\beta_1,\alpha_2\beta_2,\ldots).$$

With this definition, each $\alpha$ is of finite order,

$$\alpha^{N_\alpha} = ((1)^n;(1)^m;(1)^{n'};(1)^{m'}) = \emptyset. \tag{V.47}$$

where $N_\alpha$ is a factor of the individual N's. This operation generalizes the vertex rules for the untwisted case: if a sector $\alpha$ interacts with a sector $\beta$, it gives rise to a sector $\alpha\beta$. If each $\alpha_i,\beta_j = \pm 1$ only (the untwisted case), then this operation is equivalent to the symmetric difference operation used in the untwisted case. If a set of such vectors $\Omega = \{\emptyset,\alpha,\beta,\ldots\}$ is closed with respect to multiplication (i.e. if $\alpha\beta \in \Omega$ whenever $\alpha,\beta \in \Omega$), then $\omega$ forms an abelian group under this operation

$$\begin{aligned}
&\text{(i)} \quad \alpha\beta = \beta\alpha \\
&\text{(ii)} \quad (\alpha\beta)\gamma = \alpha(\beta\gamma) \\
&\text{(iii)} \quad \alpha\emptyset = \alpha \\
&\text{(iv)} \quad \alpha^{-1} = \alpha^{N_\alpha - 1} \in \Omega \\
&\text{(v)} \quad \alpha^{N_\alpha} = \emptyset.
\end{aligned} \tag{V.48}$$

Since, in general, an element $\alpha$ is of finite order $N_\alpha$, $\Omega$ is a finite abelian group isomorphic to a direct product of $Z_N$ factors,

$$\Omega \cong Z_{N_o} \times Z_{N_1} \times \ldots \times Z_{N_K}, \tag{V.49}$$

for some nonnegative integer K, and $\Omega$ can be generated from a set of K+1 basis elements $\{b_o,b_1,\ldots,b_K\}$, of orders $N_o,N_1,\ldots,N_K$ respectively. The sectors are given by

$$\Omega = \{ \prod_{i=0}^{K} b_i^{n_i} : n_i \in Z,\ 0 \leq n_i \leq N_i - 1\}. \tag{V.50}$$

So there are $\prod_{i=0}^{K} N_i$ sectors.

The projections contributing to $P_{\alpha,\beta}$ are defined by requiring that the number operators of the fermions in a given sector $\beta$ have a definite parity $\varepsilon(\alpha,\beta)^*$ on the sector $\alpha$, where $^*$ means complex conjugate,

$$(-1)^{\vec{\lambda}_\beta \cdot \vec{F}}\Big|_\alpha = \varepsilon(\alpha,\beta)^*. \tag{V.51}$$

$\vec{\lambda}_\beta$ is given by the exponents of the boundary conditions of sector $\beta$, see eq. (V.45), and $\vec{F}$ is a vector whose components are the operators $F_i = \sum_{r \in Z+\Delta} :f_r^j \tilde{f}_{-r}^j:$ for complex fermions and $\sum_{s=1/2}^{\infty} b_{-s}^j b_s^j$ or $\sum_{n=1}^{\infty} d_{-n}^j d_n^j$ for real NS or R fermions, respectively; (no sum on j).

$$\vec{\lambda}_\beta \cdot \vec{F} = \sum_{j=1}^{n} \tilde{\lambda}_j \tilde{F}_j + \sum_{j=1}^{m} \tilde{\lambda}'_j \tilde{F}'_j - \sum_{j=1}^{n'} \lambda_j F_j - \sum_{j=1}^{m'} \lambda'_j F'_j , \tag{V.52}$$

where the tildes distinguish left and right movers. Then

$$P_{\alpha,\beta} = \frac{1}{N_\beta} \sum_{j=0}^{N_\beta - 1} \{(-1)^{\lambda_\beta \cdot F} \varepsilon(\alpha,\beta)\}^j , \tag{V.53a}$$

and

$$\prod_{\beta\varepsilon\Omega} P_{\alpha,\beta} = \prod_{i=0}^{K} P_{\alpha,b_i} = \left( \prod_{i=0}^{K} \frac{1}{N_i} \right) \sum_{\beta\varepsilon\Omega} \varepsilon(\alpha,\beta)(-1)^{\lambda_\beta \cdot F} , \tag{V.53b}$$

where $P^2_{\alpha,\beta} = P_{\alpha,\beta}$. Since

$$(-1)^{\lambda_\beta \cdot F} (-1)^{\lambda_\gamma \cdot F} = (-1)^{\lambda_{\beta\gamma} \cdot F} , \tag{V.54a}$$

it follows that

$$\varepsilon(\alpha,\beta)\varepsilon(\alpha,\gamma) = \varepsilon(\alpha,\beta\gamma) \tag{V.54b}$$

and $\varepsilon(\alpha,\emptyset) = 1$. $\varepsilon(\alpha,\beta)$ is an $N_\beta^{th}$-root of unity. As in the untwisted case. (V.54b) is true provided that the definitions of the parities of the number operators in (V.54a) agree on the vacuum states. Using (V.53), we can express eq. (V.2) as

$$\Lambda_f = \left(\prod_{i=0}^{K} \frac{1}{N_i}\right) \sum_{\alpha\in\Omega} \sum_{\beta\in\Omega} \delta_\alpha \epsilon(\alpha,\beta)\, \mathrm{tr}_\alpha\{\bar{w}^{\bar{L}_0 - \frac{1}{2}} w^{L_0 - \frac{1}{2}} (-1)^{\lambda_\beta\cdot F}\}. \tag{V.55}$$

Again denoting the trace by $\{\alpha,\beta\}$ and leaving aside $\Pi\, N_i^{-1}$, (V.55) becomes

$$\sum_{\alpha\in\Omega} \sum_{\beta\in\Omega} \delta_\alpha \epsilon(\alpha,\beta)\{\alpha,\beta\} \tag{V.56}$$

which must transform in such a way as to keep the full partition function modular invariant. Here

$$\begin{aligned}\{\alpha,\beta\} &= \mathrm{tr}_\alpha\{\bar{w}^{\bar{L}_0 - \frac{1}{2}} w^{L_0 - \frac{1}{2}} (-1)^{\lambda_\beta\cdot F}\}\\ &= |w|^{-1}|f(w)|^{-20} \prod_{i=1}^{n} [\bar{\Theta}\left(\begin{smallmatrix}\tilde{\lambda}^i_\alpha\\ \tilde{\lambda}^i_\beta\end{smallmatrix}\right)(0|\tau)]^{1/2} \prod_{i=1}^{m} [\bar{\Theta}\left(\begin{smallmatrix}\tilde{\lambda}^i_\alpha\\ \tilde{\lambda}^i_\beta\end{smallmatrix}\right)(0|\tau)]\\ &\quad \prod_{i=1}^{n'} [\Theta\left(\begin{smallmatrix}\lambda^i_\alpha\\ \lambda^i_\beta\end{smallmatrix}\right)(0|\tau)]^{1/2} \prod_{i=1}^{m'} [\Theta\left(\begin{smallmatrix}\lambda^i_\alpha\\ \lambda^i_\beta\end{smallmatrix}\right)(0|\tau)]\ .\end{aligned} \tag{V.57}$$

As in (V.20a,b), we find from $\tau\to\tau+1$ and $\tau\to\frac{-1}{\tau}$, respectively, that

$$\epsilon(\alpha,\beta) = e^{-i\frac{\pi}{4}\lambda_\alpha\cdot\lambda_\alpha}\epsilon(\alpha,\bar{\alpha}^{-1}\beta) \tag{V.58a}$$

$$\epsilon(\alpha,\beta) = e^{i\frac{\pi}{2}\lambda_\alpha\cdot\lambda_\beta}\delta_\alpha\delta_\beta\epsilon(\beta,\alpha)^* \tag{V.58b}$$

where the scalar product is defined as

$$\lambda_\alpha \cdot \lambda_\beta = \frac{1}{2} \sum_{j=1}^{n} \tilde{\lambda}_j^{(\alpha)} \tilde{\lambda}_j^{(\beta)} + \sum_{j=1}^{m} \tilde{\lambda}'^{(\alpha)}_j \tilde{\lambda}'^{(\beta)}_j - \frac{1}{2} \sum_{j=1}^{n'} \lambda_j^{(\alpha)} \lambda_j^{(\beta)} - \sum_{j=1}^{m'} \lambda'^{(\alpha)}_j \lambda'^{(\beta)}_j \qquad (V.59)$$

Consistency of (V.58) requires:

$$N_i \lambda_{b_i} \cdot \lambda_{b_i} = 0 \pmod 8), \qquad (V.60)$$

if $N_i$ is even, and

$$N_{ij} \lambda_{b_i} \cdot \lambda_{b_j} = 0 \pmod 4), \qquad (V.61)$$

where $N_{ij}$ is the least common multiple of $N_i$ and $N_j$. For the consistency of (V.58a,b) with (V.64) below, we also require the number of real fermions that are simultaneously periodic in any four sectors to be even.

In summary, a twisted fermion string theory is defined for a given nonnegative integer, K, by a choice of K+1 generator sectors $b_0, b_1, \ldots, b_K$, together with the values of $\frac{1}{2}K(K+1) + 1$ projections $\eta_{ij} = \delta_{b_i} \epsilon(b_j, b_i)^*$ for $i=j=0$ and $0 \leq j < i \leq K$. In the general case, we choose $b_0 = F$. The remaining $\eta_{ij}$ are then obtained from:

$$\eta_{ij} = e^{\frac{i\pi}{2} \lambda_{b_i} \cdot \lambda_{b_j}} \eta_{ji}^* \qquad (V.62)$$

and

$$\eta_{oi} = \delta_{b_i} \, e^{\frac{-i\pi}{4}\lambda_{b_i}\cdot\lambda_{b_j}} \eta^*_{ii}. \tag{V.63}$$

The complete projections $\epsilon(\alpha,\beta)$, for $\alpha = \prod_{i\epsilon I} b_i$ and $\beta = \prod_{j\epsilon J} b_j$, are then given by

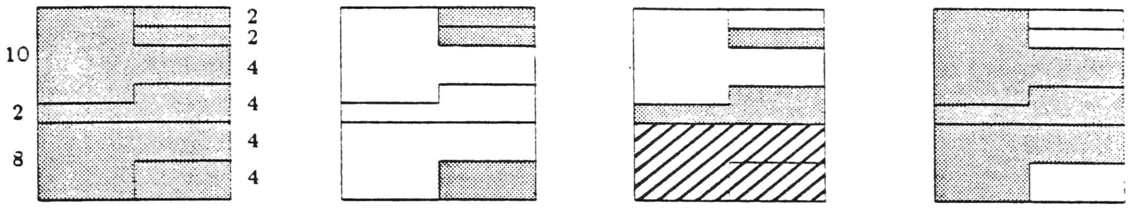
$$\epsilon(\alpha,\beta) = \delta_\beta \prod_{\substack{i\epsilon I \\ j\epsilon J}} \eta_{ij} e^{\frac{i\pi}{2}\lambda_\alpha\cdot\lambda_{b_j}} . \tag{V.64}$$

Lastly, in every sector $b_i$, the diagonal matrix $w^a_b$ defined by the boundary conditions $f^a(e^{2\pi i}z) = w^a_b f^b(z)$ must be such that $w^a_b\delta_{b_i}$ is an automorphism of the Lie algebra used to define the super-Virasoro generators.[59]

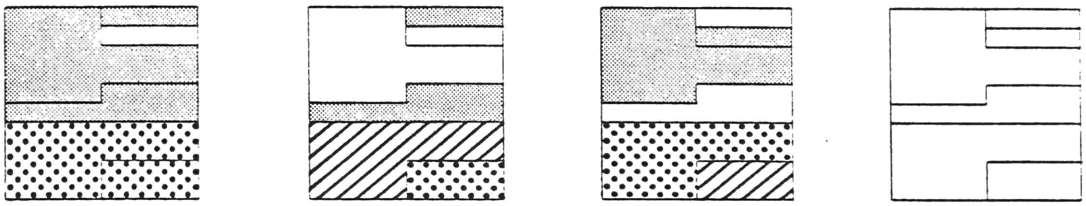
We now give an example of a chiral model in 4 dimensions.

EXAMPLE 4

An N=1 chiral model with $SU(2)\times U(1)^5$ gauge symmetry[33] can be written in terms of three generators, $b_o, b_1, b_2$; $N_o=N_1=2$, $N_2=4$; K=2. $\Omega \cong Z_2\times Z_2\times Z_4$. There are 16 sectors whose fermion content can be labelled by the box diagrams on the next page.

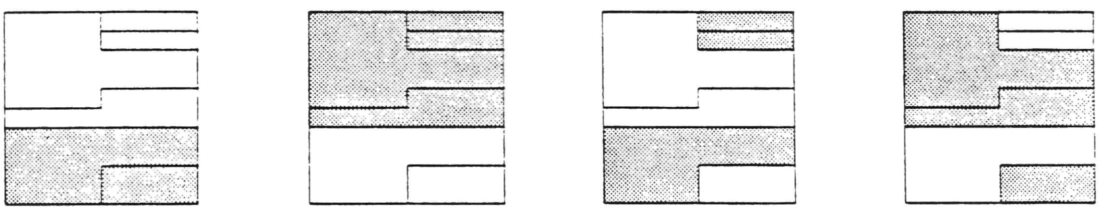



$b_0$ $b_1$ $b_2$ $b_0 b_1$



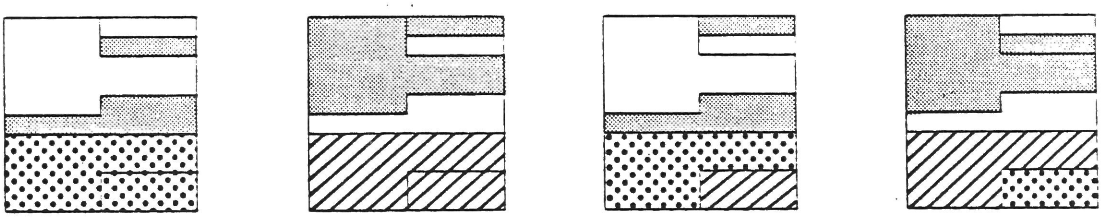
$b_0 b_2$ $b_1 b_2$ $b_0 b_1 b_2$ $\emptyset$

*(singly twisted)*



$b_2{}^2$ $b_0 b_2{}^2$ $b_1 b_2{}^2$ $b_0 b_1 b_2{}^2$

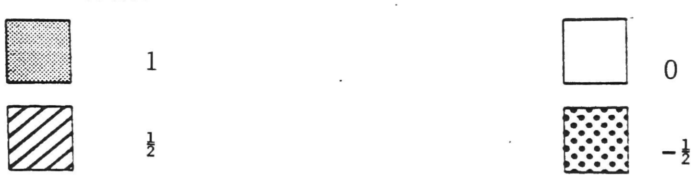
*(doubly twisted)*



$b_2{}^3$ $b_0 b_2{}^3$ $b_1 b_2{}^3$ $b_0 b_1 b_2{}^3$

*(anti-twisted)*

The boxes are now labelled by four shadings. corresponding to the boundary conditions on the four fermions:



The basis vectors are given by

$$\lambda_{b_0} = ((1)^{12};(1)^4;(1)^{12};(1)^4)$$
$$\lambda_{b_1} = ((0)^{12};(0)^4;(1)^4(0)^8;(0)^2(1)^2)$$
$$\lambda_{b_2} = ((0)^{10}(1)^2;(\tfrac{1}{2})^4;(0)^2(1)^2(0)^4(1)^4;(\tfrac{1}{2})^4).$$

The massless states come from the sectors: $b_1$, $b_2$, $\emptyset$, $b_1b_2$, $b_2^2$, $b_1b_2^2$, $b_2^2$, $b_1b_2^2$, $b_2^3$, $b_1b_2^3$. For $g = SU(2)\times U(1)^5$, they are: (The lines denote a sum of permuted entries).

1) from the untwisted sector,

| | |
|---|---|
| spin $(\pm 2,\pm\frac{3}{2})$ | in $(1;0,0,0,0,0)$ of $g$ |
| $(\pm 1,\pm\frac{1}{2})$ | in adjoint of $g$ |
| $(\frac{1}{2},0)$ | in $(1;\pm 1,0,0,0,0) \oplus 2(1;0,\underline{1,0,0,0})$ of $g$ |
| $(\frac{-1}{2},0)$ | in $(1;\pm 1,0,0,0,0) \oplus 2(1;0,\underline{-1,0,0,0})$ of $g$ |
| $(\pm\frac{1}{2},2(0))$ | in $(1;0,0,0,0,0)$ of $g$ |

2) from the singly twisted sector,

$(\frac{1}{2},0)$ in $2(1;\pm\frac{1}{2},\underline{\frac{-3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}})$ of $g$

3) from the anti-twisted sector,

$(\frac{-1}{2},0)$ in $2(1;\pm\frac{1}{2},\underline{\frac{3}{4},\frac{-1}{4},\frac{-1}{4},\frac{-1}{4}})$ of $g$

4) from the doubly twisted sector,

$(\frac{1}{2},0)$ in $(1;0,\underline{\frac{1}{2},\frac{1}{2},\frac{-1}{2},\frac{-1}{2}}) \oplus (1;0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$
$\oplus (1;0,\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}) \oplus 2(1;0,\underline{\frac{-1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}})$ of $g$

$(\frac{-1}{2},0)$ in $(1;0,\underline{\frac{-1}{2},\frac{-1}{2},\frac{1}{2},\frac{1}{2}}) \oplus (1;0,\frac{-1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2})$
$\oplus (1;0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}) \oplus 2(1;0,\underline{\frac{1}{2},\frac{-1}{2},\frac{-1}{2},\frac{-1}{2}})$ of $g$.

## Conclusion

Four-dimensional superstrings continue to be a current topic of investigation in string theory, and several interesting ideas have developed out of the results presented in this thesis. John Schwarz, in a recent review,[9] discusses some possibilities for modifying the type II theories[1,2] to obtain a less restrictive fermion spectrum. One of the ideas mentioned is to introduce two-dimensional gauge fields as additional dynamical degrees of freedom on the worldsheet.[61] This could possibly create a larger value of $\hat{c}$ for the internal space. Another question that remains is how to break spacetime supersymmetry - it is not yet clear how to do this, and the topic is still being investigated.

The new techniques for compactification and introducing symmetry into string theory as presented in this thesis have done for the type II superstring what the FKS construction did for the heterotic string. Four-dimensional type II superstrings may now be considered as promising candidates for a phenomenologically realistic theory on an equal footing with the heterotic string. Type II models have only a small number of gauge groups, which are of the right size and have a reasonable number of massless fermions; whereas the heterotic string has large groups and many more massless fermions. The work presented in this thesis has taken three major steps towards our ultimate goal of formulating a string theory directly in four dimensions with the gauge group $SU(3)\times SU(2)\times U(1)$ and the right number of chiral fermions: 1) we found a way to introduce nonabelian gauge symmetry into compactified type II strings (which had been thought to be impossible), and we developed a

superstring analog of the FKS construction.[1] 2) a new method of introducing symmetry into string theory was used to create two new models directly in four dimensions having rank-4 gauge groups that contain the gauge group of the standard model as a subgroup.[2] 3) chiral four-dimensional models were investigated, and it was shown how the requirements of unitarity and one-loop modular invariance lead to the same constraints as imposed by mult-loop modular invariance.[3]

There is still much to learn about the fundamentals of string theory, but we remain optimistic that the ultimate goal of attaining an understanding of all the elementary particle interactions in terms of strings is possible.

TABLE OF ABREVIATIONS

VM - Veneziano model
FKS - Frenkel-Kac-Segal construction
NSR - Neveu-Schwarz-Ramond string (the old superstring)
GSO - Gliozzi, Scherk, and Olive (projections on the NSR string)
VS - Virasoro-Shapiro model (closed bosonic string)
GGRT - Goddard, Goldstone, Rebbi, and Thorn, ref. [40]
OPE - Operator product expansion

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