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Infinite Sets of Conserved Charges and Duality in Quantum Field Theory

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INFINITE SETS OF CONSERVED
CHARGES AND DUALITY IN
QUANTUM FIELD THEORY

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to Carolyn

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I. Introduction

The concept of an infinite set of conserved charges in 1+1 dimensional quantum field theories has received considerable attention. Such sets have been found both for continuum theories such as the sine-Gordon equation[1] and lattice theories such as the XYZ spin chain.[2] The presence of an infinite set of conserved charges it closely related to the solution of the theory by the inverse scattering method.[3] Knowledge of the form of the conserved charges can be of value in determining how to implement an inverse scattering solution to a given theory. It has also been shown, in many cases, that the existence of an infinite set of local conserved charges implies the lack of particle production in scattering.[4] For interacting field theories such as those mentioned above (the XYZ spin chain model has been shown to have the massive Thirring model as its continuum limit[5]) this implies a much more restricted form of the S-matrix than would be expected for such theories. Clearly therefore, as has been the case with so many other symmetries and conservation laws, infinite sets of conserved charges provide a great deal of useful information about a theory.

Of course, in the physics of the real world one is not interested in theories in only two dimensions but rather in four dimensions. In four dimensions the property of possessing an infinite set of conserved charges appears at first to be too strong to be useful. It has been shown in 3+1 dimensions that theories which possess this property along with several other reasonable requirements are necessarily free theories, provided the charges are constructed from local currents.[28] Such theories are therefore uninteresting. The possibility is left open, however, for the case of non-local charges. It has been speculated that the Yang-Mills equations in loop space, which are similar to the equations of the two-dimensional chiral models, many possess charges of this sort.[6,7] Polyakov has shown how one might construct conserved charges for the loop Yang Mills theory in 2+1 dimensions in analogy with the conserved charges of the chiral model.[7] The charges have been formulated, however they are in terms of a generat-

ing function which is given as a solution to a loop space differential equation which has yet to be solved. Nevertheless, their existence is a distinct possibility. For the case of four dimensions there has been no specific form proposed for conserved charges of this type. If they were to be found they could provide valuable non-perturbative information toward a solution of the Yang-Mills system, thought to be the theory describing the strong interactions of hadronic physics. The principal long range motivation for the work described here is the search for additional symmetry in the Yang-Mills equations of the sort which leads to an infinite set of conserved charges as described above.

Kramers-Wannier self-duality[8,9] is a property of certain classical statistical mechanical systems. Through use of the transfer matrix formalism, one can find a quantum Hamiltonian associated with the statistical theory which operates on a spatial manifold of one fewer dimension.[10,11] The self-duality property has an analogue in the associated quantum theory, which takes the form of a mapping between the large and small coupling behavior of the theory. This can be useful in solving the theory by approximation techniques, e.g. if a low coupling perturbation expansion exists for the theory, then its validity can be extended to the strong coupling region by the duality transformation (Of course the intermediate coupling region may still be inaccessible through this approach). It is known that Z(N) spin theories in two dimensions (of which the Z(2) theory is the familar Ising model) are self-dual as are the corresponding Z(N) gauge theories in four dimensions.[12] What is not known is whether the SU(N) theory is also self-dual. A duality transformation for the SU(N) gauge theory has been widely sought since it could help unlock the mystery of the strong coupling, perhaps confining, behavior of the Yang-Mills theory. Such a transformation, however, has proved elusive. A partial success has been achieved by 't Hooft and others in which only the Z(N) invariant subgroup of SU(N) is subjected to the dual transformation.[13,14] Interestingly, it is the same set of variables which occur in this formulation of a dual transformation and in Polyakov's work involving conserved charges for the gauge theory. These are the path ordered expon-

entials of the gauge field around loops:

$$A(C) = \frac{1}{2}\,\mathrm{tr}\,P\,e^{i \oint_C \vec{A}\cdot d\vec{x}} \tag{1.1}$$

This suggested a question which has been partially answered by the work herein: what, if any, is the relationship between the properties of self-duality and complete integrability? In particular, can self-duality aid one in the search for conserved charges, or conversely does knowledge of the conserved charges lead to a dual transformation?

To the first half of this question I can give a qualified yes which is the main result of this thesis. Specifically I will show that any self-dual quantum theory linear in the coupling constant possesses an infinite set of conserved commuting charges provided one additional condition is met, which is essentially that the first charge in the set is conserved. The set is defined by an iterative commutation procedure involving the operators in the Hamiltonian. This theorem is shown to be applicable to a number of self-dual spin theories: the Ising model in a transverse magnetic field, the XY spin chain model, and the extended XY model of Suzuki.[27] For these theories it has the advantage of providing the explicit form of the higher charges with much less algebra than would be necessary if one were to use either the tranfer matrix or inverse scattering method. This may be of use in studying the relevance of the higher charges. Unfortunately, however, the theorem does not seem to apply to either the two-dimensional Z(N) spin system or the four-dimensional Z(2) gauge theory. Thus the prospect appears rather bleak that it will prove useful for the case of the SU(N) gauge theory, although one cannot be certain until a properly self-dual formulation of that theory is found.

The thesis consists of nine chapters and two appendices. In chapter II I discuss the connection given by the transfer matrix formulation between a classical statistical mechanical system and an associated quantum Hamiltonian, using as an example the familiar two-

dimensional Ising model. Next, in chapter III, I show how this applies to Baxter's 8-vertex model and the XYZ Hamiltonian, and how an infinite set of conserved charges for the XYZ model arises from properties of the transfer matrix. In Chapter IV I discuss the meaning of self-duality as it applies to quantum Hamiltonians, and develop a general form for a self-dual Hamiltonian. In Chapter V I show how one can find the explicit form for conserved charges in the XY model by a heuristic technique which takes advantage of the models' self-duality. In chapter VI it is shown how the solution to the XYZ model by the quantum inverse scattering method leads to the infinite set of conserved charges associated with it. The thesis culminates in the statement and proof of the theorem mentioned above, which stipulates a condition under which self-dual theories possess an infinite set of conserved charges (Chapter VII), after which I briefly discuss applications of the theorem (VIII) and draw some conclusions (IX).

II. Classical Statistical Mechanics and Quantum Field Theory

2.1 Relation Through the Transfer Matrix

The Euclidean path integral for a quantum field theory in N dimensions is given by

$$Z_T(J) = \mathcal{N}^{-1}\int D\varphi \exp\left[-\frac{1}{\hbar}\int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} d^{N-1}x \left(\mathcal{L}_E(\varphi(x,t)) + J(x,t)\varphi(x,t)\right)\right] \tag{2.1}$$

where $\mathcal{L}_E$ is the Euclidean Lagrangian density, $\varphi(x,t)$ is the field to be quantized, and $J(x,t)$ is an external probing field. I shall consider (2.1) to have vacuum boundary conditions:

$$\varphi(x, \pm T/2) = 0 \tag{2.2}$$

The normalizing factor $\mathcal{N}$ is chosen so that

$$Z_T(0) = 1 \tag{2.3}$$

In the limit $T \to \infty$, $Z_T(J)$ is the generating functional for the Euclidean Green's functions of the theory. One can also interpret (2.1) directly as the vacuum to vacuum transition amplitude over an imaginary time iT in the presence of an external field J. In the Schrodinger picture,

$${}_{T/2}\langle 0| e^{-\hat{H}T} |0\rangle_{-T/2} = Z_T(J) \tag{2.4}$$

Equation (2.1) looks very much like the partition function of an N-dimensional statistical mechanical system. If one defines a classical Hamiltonian for a corresponding statistical mechanical theory

$$H_c = \int_{-T/2}^{T/2} dt \int_{-\infty}^{\infty} d^{N-1}x \left(\mathcal{L}_E(\varphi(x,t)) + J(x,t)\,\varphi(x,t)\right) \tag{2.5}$$

and an inverse temperature

$$\beta^{-1} \equiv k(T_{emp.}) = \hbar \tag{2.6}$$

then

$$Z_T(J) = \mathcal{N}^{-1}\int D\varphi \exp\left[-\beta H_c\right] \tag{2.7}$$

which, with the exception of the boundary condition at $\pm T$, has the form of a normalized partition function. Note that if the original quantum system had 3+1 dimensions, then the classical Hamiltonian, H_C, would involve four spatial dimensions, so the partition functions relevant to realistic quantum field theories all have one more spatial dimension than those usually studied in statistical mechanics. Another variant from the usual case in statistical mechanics is the presence of continuous fields. Also, of course, real quantum field theory is formulated on a Minkowski space, not a Euclidean space which requires an analytic continuation of Green's functions derived from $Z_T(J)$ to imaginary time.

This connection between statistical mechanics and quantum field theory has proven extremely valuable for both fields. Quantum field theory has benefitted from simplified perturbation theory integrals, lattice field theories, Monté Carlo simulations, strong coupling expansions, and concepts such as spontaneous symmetry breaking and coupling constant phase transitions. Statistical mechanics has been given a theory of universal critical phenomena following the renormalization group. In addition, as we shall see below, the quantum connection has aided in finding the exact solutions to certain statistical theories. This list is by no means complete. It is being added to continuously.

The transfer matrix formulation provides a convenient method to determine the associated quantum Hamiltonian as occurs in (2.4) for a given statistical system (2.7). Consider a statistical system defined on a lattice of points (fig. 1). For the sake of clarity I shall consider a two-dimensional lattice although the method works for any number of dimensions. At each point there exists a classical field $\mathcal{Q}(i,j)$ which could take on either a discrete or continuous set of values. Now construct a Hilbert space of states at each point which are eigenstates of a quantum field $\mathcal{Q}(i,j)$ whose eigenvalues are in one-to-one correspondence to the set of allowed values for the classical field. Form a state vector for a horizontal row of points by taking

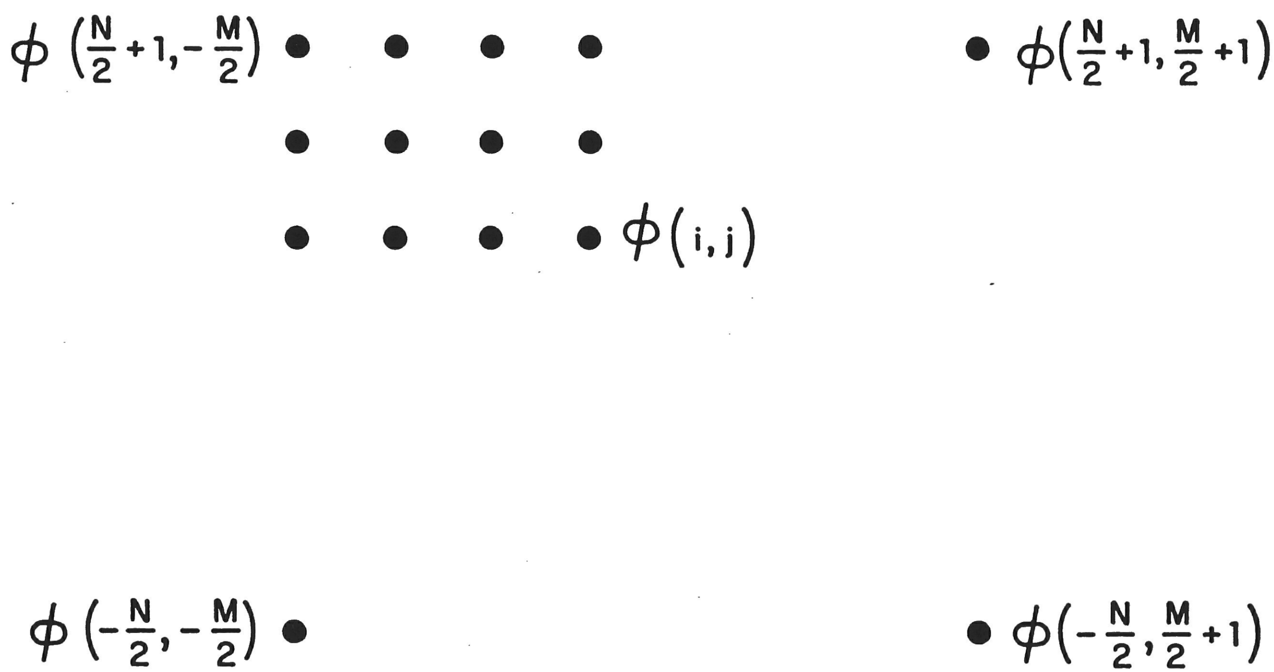

Fig. 1. A general two-dimensional lattice for statistical mechanics.

the direct product of states for those points.

$$|\vec{\varphi}(i)\rangle = |\varphi(i,-M/2)\rangle \otimes |\varphi(i,-M/2+1)\rangle \otimes \cdots \otimes |\varphi(i, M/2+1)\rangle \qquad (2.8)$$

Then one attempts to find a matrix $\hat{T}$ whose elements give the multiplicative contribution to the partition function from two adjacent horizontal rows in a particular configuration (Note that this is only possible if the interactions in the vertical direction are no more extended than nearest neighbor). The partition function will then be given by

$$Z_T = \sum_{\substack{|\vec{\varphi}(i)\rangle \\ -\frac{N}{2}+1 \le i \le N/2}} \langle \vec{\varphi}(\tfrac{N}{2}+1)|\hat{T}|\vec{\varphi}(N/2)\rangle\langle\vec{\varphi}(N/2)|\hat{T}|\vec{\varphi}(N/2-1)\rangle \cdots \times \langle \vec{\varphi}(-\tfrac{N}{2}+1)|\hat{T}|\vec{\varphi}(-N/2)\rangle \,. \qquad (2.9)$$

Since the intermediate sums are over complete sets of states,

$$Z_T = \langle \vec{\varphi}(\tfrac{N}{2}+1)|\hat{T}^N|\vec{\varphi}(-N/2)\rangle \qquad (2.10)$$

If one drops the boundary condition, identifies the rows i=N/2+1 and i=-N/2 and sums over that state also then one obtains the usual partition function with periodic boundary conditions:

$$Z = \mathrm{tr}\, \hat{T}^N \qquad (2.11)$$

Comparing (2.10) with (2.4) and realizing that one could have taken any boundary conditions, not just vacuum boundary conditions, leads to the identification of the operators

$$\hat{T}^N = e^{-\hat{H}T} \qquad (2.12)$$

or

$$\hat{T} = e^{-\hat{H}(T/N)} = e^{-\hat{H}\tau} \,. \qquad (2.13)$$

τ is the lattice spacing in the vertical direction which can be thought of as the time direction. Thus the transfer matrix is seen to be the time evolution operator for a single lattice step in a Euclidean lattice version of an associated quantum theory.

The transfer matrix approach has been useful in the solution of partition functions. For an infinite lattice system ($N\to\infty$, $T\to\infty$)

$$Z = \mathrm{tr}\,\hat{T}^N \Rightarrow T_{max}^N \tag{2.14}$$

where T_{max} is the maximum eigenvalue of $\hat{T}$. This is true provided T_{max} is unique. Other contributions are exponentially small as $N\to\infty$. As an aside, note also that

$$Z = T_{max}^N = e^{-E_0 \tau N} = \langle 0| \hat{T}^N |0\rangle = Z_T \tag{2.15}$$

where E_0 is the ground state energy of H. Thus in the limit $T\to\infty$ it doesn't matter whether we work with Z_T or Z. If T_{max} is not unique, however, the boundary conditions may be important even in this limit.

Thus the problem of solving the partition function is reduced to finding the maximum eigenvalue of the transfer matrix. This is not necessarily simple, however, since the matrix $\hat{T}$ is often very complicated. One approach has been to find a simple quantum Hamiltonian, not necessarily the same as $\hat{H}$ above, which commutes with $\hat{T}$.[15] Then a transformation which diagonalizes the simpler quantum Hamiltonian which in some cases might already be known, would greatly simplify the diagonalization of $\hat{T}$ since they share a common set of eigenvectors. In the limit $\tau\to 0$ the Hamiltonian $\hat{H}$ often becomes quite simple and one can solve for Z by computing the ground state energy of $\hat{H}$. This gives Z only for an extremely anisotropic lattice ($\tau\to 0$). However, it is still a useful connection since some properties of the system such as its critical exponents are independent of the lattice spacing. This so-called τ-continuum limit will be discussed more thoroughly in the next section.

2.2 The Ising Model

As an example of the methods discussed above, consider the case of the two-dimensional anisotropic Ising model.[16,17] Its partition function is given by

$$Z = \sum_{S_{i,j}=\pm 1} \exp\left[\sum_{i,j=1}^{N,M} \left(\beta_\tau S_{i,j} S_{i+1,j} + \beta S_{i,j} S_{i,j+1}\right)\right] . \tag{2.16}$$

The model is defined on a periodic NxM lattice of points upon each of which resides a two valued variable $S_{i,j}=\pm 1$ (see fig. 2). The couplings in the vertical and horizontal directions are given by β_τ and β respectively. To find a transfer matrix for this model consider the states

$$|\vec{S}_i\rangle \equiv |S_{i,1}\rangle \otimes |S_{i,2}\rangle \otimes |S_{i,3}\rangle \otimes \cdots \otimes |S_{i,M}\rangle \tag{2.17}$$

where

$$|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad |-1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.18}$$

are Pauli spinors. The states $|\vec{S}_i\rangle$ defined in (2.17) are in one-to-one correspondence with the possible classical configurations of the variables $\{S_{ij}\}$ in that row. The goal is to find an operator $\hat{T}$ such that

$$Z = \mathrm{tr}\,\hat{T}^N = \sum_{|\vec{S}_i\rangle} \langle \vec{S}_N|\hat{T}|\vec{S}_{N-1}\rangle\langle \vec{S}_{N-1}|\hat{T}|\vec{S}_{N-2}\rangle \cdots \langle \vec{S}_1|\hat{T}|\vec{S}_N\rangle . \tag{2.19}$$

Consider first only the contribution to Z from two adjacent temporal (vertical) points in column J. Expressed as a matrix which can be sandwiched between the spinors representing those points, it is given by

$$M_J = \begin{pmatrix} e^{\beta_\tau} & e^{-\beta_\tau} \\ e^{-\beta_\tau} & e^{\beta_\tau} \end{pmatrix}_J . \tag{2.20}$$

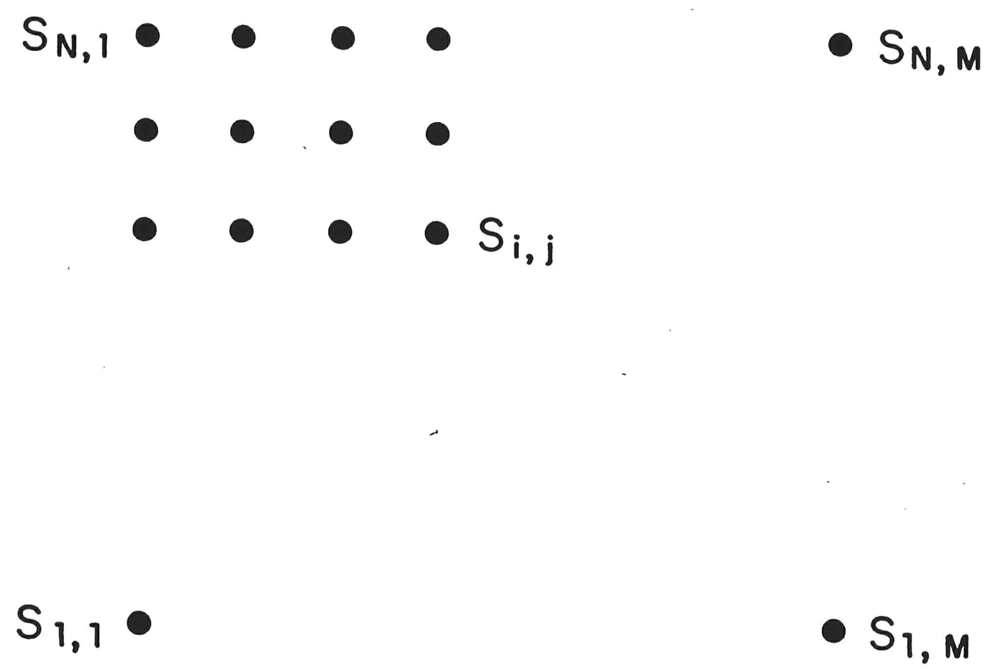

Fig. 2. Periodic lattice for the two-dimensional Ising model ($S_{i,1}=S_{i,M+1}$; $S_{1,i}=S_{N+1,i}$).

This can be expressed in terms of the Pauli spin matrix $\hat{\sigma}^1$:

$$M_J = \left(\tfrac{1}{2}\sinh(2\beta_\tau^*)\right)^{-1/2} \exp\left(\beta_\tau^* \hat{\sigma}_J^1\right) \tag{2.21}$$

where

$$\tanh \beta_\tau^* \equiv e^{-2\beta_\tau} \quad , \quad \hat{\sigma}_J^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_J \tag{2.22}$$

The contributions of temporal couplings for the whole row may be obtained by multiplying many of these matrices together. The within-row interaction is a diagonal matrix and therefore can be multiplied on the left or right without affecting the operation of the M_J. Thus

$$\hat{T} = \left(\tfrac{1}{2}\sinh(2\beta_\tau^*)\right)^{-M/2} \exp\left(\sum_J \beta_\tau^* \hat{\sigma}_J^1\right) \exp\left(\sum_J \beta\, \hat{\sigma}_J^3 \hat{\sigma}_{J+1}^3\right) \tag{2.23}$$

is an acceptable transfer matrix for (2.19). We are interested in finding a Hamiltonian which is related to $\hat{T}$ by

$$\hat{T} \propto e^{-\hat{H}\tau} . \tag{2.24}$$

However, $\hat{T}$ is the product of exponentials of operators which do not commute. A representation of it as a single exponential through a Baker-Hausdorff identity[18] results in an infinite number of terms for $\hat{H}$. Nevertheless in the τ-continuum limit[11] $\hat{H}$ becomes simple. $\hat{T}$ does not depend explicitly on τ as written, but if correlation functions for points separated by finite distances are to be kept finite, the couplings β_τ and β must be scaled appropriately as $\tau \to 0$. In addition the energy per site will become infinite in this limit and an energy renormalization will be necessary. The requirements for a properly normalized transfer matrix are 1)

$$\lim_{\tau\to 0} \hat{T}' = \lim_{\tau\to 0} e^{-\hat{H}\tau} = \hat{1} \tag{2.25}$$

and 2)

$$\lim_{\tau\to 0} -\frac{1}{\tau} \ln(\hat{T}') = \hat{H} \tag{2.26}$$

where $\hat{H}$ is some <u>finite</u> operator. This can be accomplished by scaling β_τ^* and β such that

$$\beta_\tau^* \propto \tau \,, \quad \beta \propto \tau \tag{2.27}$$

and, for the energy renormalization, simply removing the numerical prefactor from $\hat{T}$ (The latter could also have been accomplished by taking the original temporal interaction in (2.16) to be $-\frac{1}{2}\beta_\tau(S_{i,j}-S_{i+1,j})^2$ instead of $\beta_\tau S_{i,j}S_{i+1,j}$). Thus

$$\hat{T}' = \exp\left(\sum_J \tau \hat{\sigma}_J^1\right) \exp\left(\sum_J \lambda\tau \hat{\sigma}_J^3 \hat{\sigma}_{J+1}^3\right), \tag{2.28}$$

where I have let

$$\beta_\tau^* = \tau \,, \quad \beta = \lambda\tau \,. \tag{2.29}$$

Expanding to first order in τ one obtains the τ-continuum Hamiltonian

$$\hat{H} = -\sum_J \left(\hat{\sigma}_J^1 + \lambda \hat{\sigma}_J^3 \hat{\sigma}_{J+1}^3\right) \tag{2.30}$$

which is just the Hamiltonian for a one-dimensional quantum Ising model in a transverse magnetic field. Thus we have obtained a simple one-dimensional quantum Hamiltonian related to the original two-dimensional statistical system, albeit in a very anisotropic limit. The equilibrium state of the statistical model undergoes a disordering phase transition at the point $\beta=\beta_\tau^*$. The ground state of the quantum model undergoes

an analogous non-analytic transition when the coupling λ passes through 1. The details of this transition have analogues in each model and can be studied in either, however this is not the purpose for which I shall use this connection here. In the following chapter I shall demonstrate how, for the case of the 8-vertex model, properties of the transfer matrix imply the presence of an infinite set of conserved charges for its associated quantum theory, the XYZ anisotropic Heisenberg spin chain.

III. The 8-Vertex Model and the XYZ Hamiltonian

3.1 Connection Between H_{XYZ} and T_{8V}

If one subjects the symmetric 8-vertex model of Baxter[19] to a similar treatment as that given the two-dimensional Ising model in the previous chapter, one finds that the associated τ-continuum quantum Hamiltonian is that of the XYZ anisotropic Heisenberg chain[20,21], given by

$$H_{xyz} \equiv -\frac{1}{2} \sum_{J} \left(J_x \hat{\sigma}_J^1 \hat{\sigma}_{J+1}^1 + J_y \hat{\sigma}_J^2 \hat{\sigma}_{J+1}^2 + J_z \hat{\sigma}_J^3 \hat{\sigma}_{J+1}^3 \right) \tag{3.1}$$

It will prove useful to examine this connection in some detail, since in this case known properties of the 8-vertex transfer matrix will yield some additional information on the quantum model, namely an infinite number of conserved charges. The 8-vertex model is formulated on a two-dimensional rectangular lattice of points with an arrow placed on each link (fig. 3). An energy is assigned to each point (vertex) depending upon the configuration of arrows surrounding it. Normally there would be sixteen different vertex types, but for the 8-vertex model configurations with an odd number of arrows entering or leaving a vertex are assigned an infinite energy and thereby forbidden. In the symmetric 8-vertex model of Baxter, those vertices which result from reversing the directions of all impinging arrows are deemed to have the same energy. Thus we are left with the energy assignments of fig. 4. The partition function is given by

$$Z = \sum_{\substack{\text{allowed} \\ \text{configurations}}} e^{-\beta \sum_{i=1}^{8} N_i \epsilon_i} \tag{3.2}$$

where N_i is the number of vertices of type i in a given configuration. The model lends itself to the transfer matrix approach. Since I am just reviewing the work of Baxter here, I will spare the reader the details of the derivation and simply state the results that will be useful to me. Let

$$\omega_i = \exp(-\beta \epsilon_i) \tag{3.3}$$

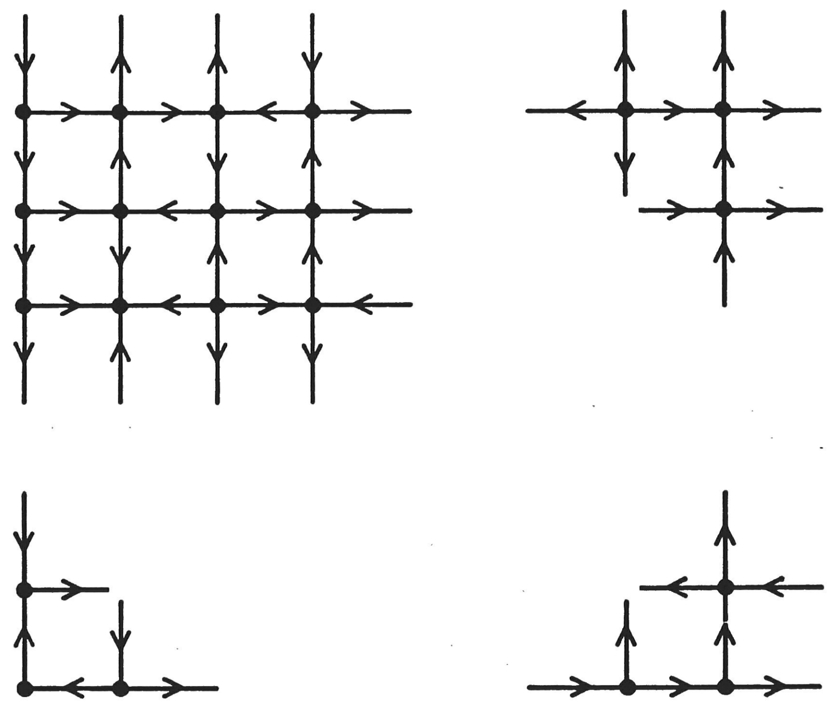

Fig. 3. 8-vertex lattice model with periodic boundary conditions.

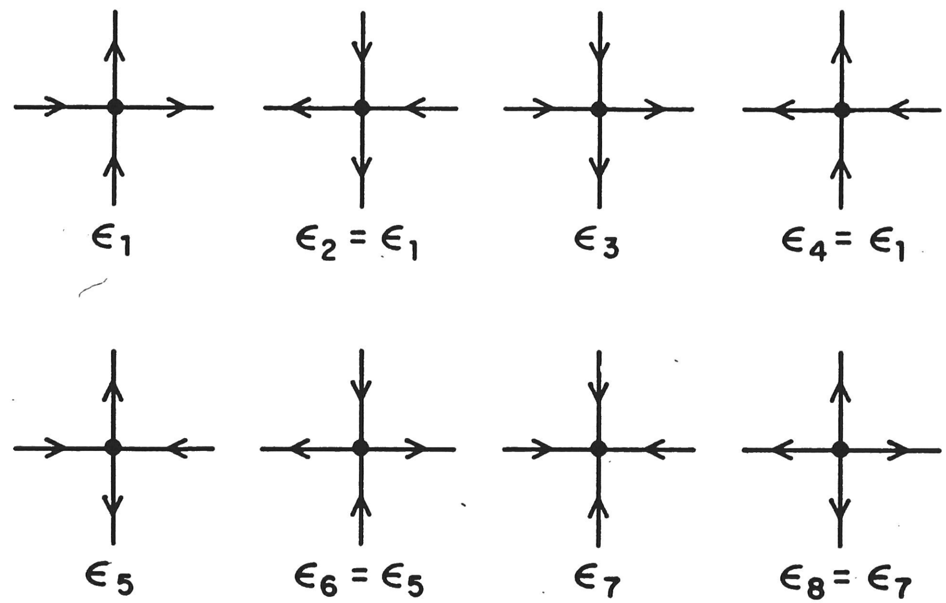

Fig. 4. Energy assignments for allowed vertices in the 8-vertex model.

and define

$$p_1 = \tfrac{1}{2}(\omega_3 + \omega_7) \;, \quad p_2 = \tfrac{1}{2}(\omega_3 - \omega_7)$$
$$p_3 = \tfrac{1}{2}(\omega_1 - \omega_5) \;, \quad p_4 = \tfrac{1}{2}(\omega_1 + \omega_5) \;. \tag{3.4}$$

Then the transfer matrix is given by

$$T_{\alpha|\alpha'} = \sum_{\lambda_1=1}^{2} \sum_{\lambda_2=1}^{2} \cdots \sum_{\lambda_M=1}^{2} \prod_{J=1}^{M} \left(\sum_{j=1}^{4} p_j \, \sigma^j_{\lambda_J \alpha'_J} \, \sigma^j_{\alpha_J \lambda_{J+1}} \right) \tag{3.5}$$

The α_J ,α'_J represent two adjacent vertical rows of spins. $\hat{T}$ gives the contribution to the partition function of all allowed configurations of the intervening horizontal row of spins, represented by λ_J. The σ matrices are the usual Pauli matrices with $\sigma^4 \equiv 1$. The structure of $\hat{T}$ is rather complicated. It is the partial matrix product of M matrices (over 2 of 4 indices) each of which is the sum of four more elementary matrices. To find a τ-continuum limit one must find a way to scale couplings with τ so that $\hat{T}\to\hat{1}$ as $\tau\to 0$ and so that $\hat{T}^{-1} \frac{\partial}{\partial\tau} \hat{T}$ is finite as $\tau\to 0$. This can clearly be done by choosing $\omega_3 \sim \tau$, $\omega_7 \sim \tau$, $(\omega_1 - \omega_5) \sim \tau$, $(\omega_1 + \omega_5) = 2 + \sim\tau$, since such a choice gives $p_1 = p_2 = p_3 \to 0$ and $p_4 \to 1$. In this limit $\hat{T}$ does not quite become a unit matrix but rather the matrix

$$\prod_J \delta_{\alpha_J \alpha'_{J+1}} \tag{3.6}$$

which is a cyclic shift operator to the left. This can be interpreted as meaning that the time direction should, in this case, be taken to be along the left leaning lattice diagonal. If one takes, in the limit $\tau\to 0$

$$p_1 \sim \alpha\tau \;, \quad p_2 \sim \beta\tau$$
$$p_3 \sim \gamma\tau \;, \quad p_4 \sim 2 + \epsilon\tau \tag{3.7}$$

Then

$$\lim_{\tau\to 0} \hat{T}^{-1}\frac{\partial}{\partial\tau}\hat{T} = \sum_J \sum_{j=1}^{4} \frac{\partial P_j}{\partial\tau}\bigg|_{\tau=0} \sigma^j_{\alpha_J\alpha'_J}\,\sigma^j_{\alpha_{J+1}\alpha'_{J+1}}$$

$$= \sum_J \alpha\,\hat{\sigma}^1_J\,\hat{\sigma}^1_{J+1} + \beta\,\hat{\sigma}^2_J\,\hat{\sigma}^2_{J+1} + \gamma\,\hat{\sigma}^3_J\,\hat{\sigma}^3_{J+1} + \text{const.}$$

$$= \hat{H}_{xyz} + \text{const.} \tag{3.8}$$

$$(\hat{\sigma}^i_J \equiv \sigma^i_{\alpha_J\alpha'_J})$$

Thus the τ-continuum quantum Hamiltonian associated with Baxter's model is seen to be the XYZ Hamiltonian.

3.2 An Infinite Set of Conserved Charges for H_{XYZ}

Baxter has discovered a very interesting property of the 8-vertex model transfer matrix, namely that there exists a one parameter set of commuting transfer matrices.[19,20] If one defines still another set of parameters w_i, i=1,2,3,4,

$$W_1 \equiv cn(V,\ell)/cn(\mathcal{J},\ell) \;, \qquad W_2 \equiv dn(V,\ell)/dn(\mathcal{J},\ell)$$
$$W_3 \equiv 1 \;, \qquad W_4 \equiv sn(V,\ell)/sn(\mathcal{J},\ell) \tag{3.9}$$

where $sn(V,\ell)$, $Cn(V,\ell)$, $dn(V,\ell)$ are the Jacobian elliptic functions of argument V and modulus ℓ, and sets

$$P_1 = \tfrac{1}{2}(W_1 - W_2 - W_3 + W_4) \;, \qquad P_2 = \tfrac{1}{2}(-W_1 + W_2 - W_3 + W_4)$$
$$P_3 = \tfrac{1}{2}(-W_1 - W_2 + W_3 + W_4) \;, \qquad P_4 = \tfrac{1}{2}(W_1 + W_2 + W_3 + W_4) \tag{3.10}$$

then

$$[\hat{T}(V,\mathcal{J},\ell), \hat{T}(V',\mathcal{J},\ell)] = 0 \quad . \tag{3.11}$$

Note that a constant energy can be added to each of the vertex energies ε_i without having any effect other than multiplying the partition function by a constant. This allows an arbitrary multiplicative scaling of the quantities w_i defined in (3.9) which Baxter has utilized to set $w_3=1$. Thus we are left with only three parameters describing the model V,ζ,ℓ.

One can see from (3.9) and (3.10) that $V\to\zeta$ is equivalent to the τ-continuum limit discussed previously (3.7). One can take, without loss of generality, $\tau=\frac{1}{k}(V-\zeta)$. Then one obtains

$$\hat{T}^{-1}\frac{\partial}{\partial\tau}\hat{T}\Big|_{\tau=0}=k\,\hat{T}^{-1}(V,\zeta,\ell)\frac{\partial}{\partial V}\hat{T}(V,\zeta,\ell)\Big|_{V=\zeta}$$
$$=k\sum_{i=1}^{4}\sum_{J}\frac{1}{2}P_i'\,\hat{\sigma}_J^i\,\hat{\sigma}_{J+1}^i=H_{xyz}+\text{const.}\tag{3.12}$$

where

$$P_i'\equiv\frac{\partial P_i(V,\zeta,\ell)}{\partial V}\Big|_{V=\zeta}\ .\tag{3.13}$$

Specifically,

$$P_1'=\frac{dn(2\zeta,\ell)}{sn(2\zeta,\ell)}\ ,\qquad P_2'=\frac{cn(2\zeta,\ell)}{sn(2\zeta,\ell)}\tag{3.14}$$
$$P_3'=\frac{1}{sn(2\zeta,\ell)}\ ,\qquad P_4'=P_1'+P_2'-P_3'\ .$$

The XYZ couplings (3.1) are given by

$$J_x=-k\,P_1'\ ,\quad J_y=-k\,P_2'\ ,\quad J_z=-k\,P_3'\ .\tag{3.15}$$

In the following we shall take k=1, since as far as conservation laws are concerned, the normalization of $\hat{H}_{XYZ}$ is not important.

As has been shown by Lüscher[2], (3.11) immediately implies the existence of an infinite set of conserved charges for the XYZ model by the following argument. One differentiates (3.11) an arbitrary number

of times with respect to V, obtaining

$$\left[\frac{\partial^N}{\partial V^N}\hat{T}(V,\zeta,\ell),\ \hat{T}(V',\zeta,\ell)\right]=0\ . \tag{3.16}$$

Next multiply on the left and the right by $\hat{T}^{-1}(V',\zeta,\ell)$.

$$\left[\hat{T}^{-1}(V',\zeta,\ell),\frac{\partial^N}{\partial V^N}\hat{T}(V,\zeta,\ell)\right]=0\ . \tag{3.17}$$

Also differentiate (3.16) once with respect to V'.

$$\left[\frac{\partial^N}{\partial V^N}\hat{T}(V,\zeta,\ell),\frac{\partial}{\partial V'}\hat{T}(V',\zeta,\ell)\right]=0\ . \tag{3.18}$$

Due to (3.17) we can multiply (3.18) by factors of $\hat{T}^{-1}(V,\zeta,\ell)$ and $\hat{T}^{-1}(V',\zeta,\ell)$, commute them through and let $V\to\zeta$ to obtain

$$\left[\hat{T}^{-1}(V,\zeta,\ell)\frac{\partial^N}{\partial V^N}\hat{T}(V,\zeta,\ell)\Big|_{V=\zeta},\ \hat{T}^{-1}(V',\zeta,\ell)\frac{\partial}{\partial V'}\hat{T}(V',\zeta,\ell)\Big|_{V'=\zeta}\right]=0\ . \tag{3.19}$$

Thus the quantity $\hat{G}_N$, defined as

$$G_N=\hat{T}^{-1}(V,\zeta,\ell)\frac{\partial^{N+1}}{\partial V^{N+1}}\hat{T}(V,\zeta,\ell)\Big|_{V=\zeta} \tag{3.20}$$

commutes with the XYZ Hamiltonian (see 3.12)

$$[\hat{G}_N,\hat{H}_{XYZ}]=0 \tag{3.21}$$

and is therefore a conserved charge for the system. It has proven useful to define another set of charges $\hat{C}_N$

$$\hat{C}_N=\frac{\partial^{N+1}}{\partial V^{N+1}}\ln\hat{T}(V,\zeta,\ell)\Big|_{V=\zeta} \tag{3.22}$$

which are related to the $\hat{G}_N$ in exactly the same way as connected Green's functions are to complete Green's functions in field theory. Thus

$$\begin{aligned} \hat{C}_0 &= \hat{G}_0 = \hat{H}_{XYZ} + \text{const.} \\ \hat{C}_1 &= \hat{G}_1 - \hat{G}_0^2 \\ \hat{C}_2 &= \hat{G}_2 - 3\hat{G}_1\hat{G}_0 + 2\hat{G}_0^3 \ . \end{aligned} \tag{3.23}$$

The advantage of the $\hat{C}_N$ is that they are expressable as sums over <u>local</u> current densities. Such charges are called local charges although the charges themselves are not local operators. A local charge symmetry can have a strong effect on particle dynamics since it results in an additively conserved quantum number for the particles. (e.g. in the continuum version of the XYZ model - the massive Thirring model - the conserved charges $\hat{C}_N$ imply $\sum_i p_i^{N+1}$ = constant for any assembly of particles $\{i\}$ with momenta $\{p_i\}$ [2]). It is interesting to note that any Hermitian Hamiltonian system possesses an infinite set of conserved charges, e.g. the projection operators onto energy eigenstates $\hat{Q}_n = |E_n\rangle\langle E_n|$ are independent conserved operators. They are not, however, in general local. Thus the issue is not whether a system possesses an infinite set of conserved charges but rather whether the forms of such charges can be found in terms of explicitly known operators and whether or not the charges are local. A non-local set of charges does not generally inhibit particle dynamics to the extent that local charges do, however it may still be as effective in aiding the integration of the theory. It is this scenario which is hoped to be active in the four-dimensional gauge theory.

The first charge for the XYZ model, $\hat{C}_1$, can be obtained rather easily. The differentiation of (3.5) twice with respect to V leaves three types of terms: one in which the two derivatives fall on the same factors of p_i, one in which they fall upon factors of p_i associated with matrices from adjacent sites linked by matrix multiplication, and one in which the derivatives fall upon factors of p_i associated

with matrices from more distant sites. We shall call these double overlapping, single overlapping and non-overlapping terms respectively. Computation shows that the $-\hat{G}_o^2$ term in (3.23) subtracts off the double overlapping and non-overlapping terms, as well as the symmetric part of the single overlapping term, which leaves

$$\begin{aligned}\hat{C}_1 &= \frac{1}{4} \sum_J \sum_{i,j=1}^{4} P_i' P_j' \, \sigma_J^i \, [\sigma_{J+1}^i, \sigma_{J+1}^j] \, \sigma_{J+2}^j + \text{const.} \\ &= \frac{i}{2} \sum_J \sum_{i,j=1}^{3} P_i' P_j' \, \epsilon_{ijk} \, \sigma_J^i \, \sigma_{J+1}^k \, \sigma_{J+2}^j + \text{const.}\end{aligned} \tag{3.24}$$

where

$$P_i' = \frac{\partial}{\partial V} P_i \Big|_{V=J} \tag{3.25}$$

Computing the explicit forms of the charges beyond $\hat{C}_1$ presents a lengthy algebraic task. In Chap. V, a heuristic procedure will be used to determine the explicit form of the N^{th} charge for the simpler XZ model ($J_y=0$). In Appendix A the result of a direct calculation of $\hat{C}_2$ for both the XYZ and XZ models is given, primarily as a check on the procedure of Chap. V. In contrast the theorem of Chap. VII will provide a much simpler method for finding the XZ model charges, as well as charges for any other model to which it applies.

IV. Self-Duality in Quantum Theories

Kramers-Wannier duality and self-duality[8,9] are properties of certain classical statistical mechanical models which have proven useful in the investigation of critical phenomena, the characterization of the behavior of models at temperature extremes, and in the discovery of relationships between different models. Not surprisingly, the presence of a dual transformation in a given statistical theory is reflected in an analogous property of the associated quantum Hamiltonian theory. Self-duality in a quantum system, as we shall see, takes the form of a change of variables whose effect is to switch the "free" and "interaction" parts of the Hamiltonian. The transformation therefore relates two versions of the same theory, one with weak coupling and the other with strong coupling. In this chapter I will review self-duality in statistical theories, demonstrate how this leads to a self-duality property for the associated quantum Hamiltonian, using the Ising model as an example, and determine the most general form that a self-dual quantum Hamiltonian can take.

4.1 Self-Duality in Statistical Mechanics

Say one is given a partition function for some statistical system described by variables s_i:

$$Z = \sum_{s_i} e^{-\beta H(s_i)} \tag{4.1}$$

In some cases one can find a new set of variables u_i in which the same partition function is given in terms of a new Hamiltonian $\tilde{H}(u_i)$ and a new temperature $\bar{\beta}(\beta)$.

$$Z = \sum_{u_i} e^{-\bar{\beta}\tilde{H}(u_i)} \tag{4.2}$$

A definite procedure must be given by which to find the variables u_i, and it must be such that applying the procedure a second time leads back to the original variables s_i, thus the name duality. The relationship between $\{s_i\}$ and $\{u_i\}$ is usually not a direct change of

variables $s_i=f(\{u_i\})$ which would imply a correspondence between terms of (4.1) and (4.2). Rather it is generally a much weaker connection such that only the sums over all configurations are equal, with each term in the s_i sum having contributions from many terms in the u_i sum and vice-versa. This "weakness" is actually the source of much of the power behind the dual transformation; e.g. consider a system which is highly disordered in the variables s_i. One would expect a large number of terms to contribute in a substantial way to the sum (4.1). However the dual variables u_i are usually such that they measure the degree of disorder in the original system, thus one would expect only a relativity few terms to contribute substantially to the sum (4.2) giving very probably a simpler description of the highly disordered state.

In the case that

$$\tilde{H}(u_i) = H(u_i) \tag{4.3}$$

the model is said to be self-dual. In this case the duality requirement implies

$$\bar{\bar{\beta}} = \beta \tag{4.4}$$

i.e. the function $\bar{\beta}(\beta)$ is its own inverse.

$$\bar{\beta}(\bar{\beta}(\beta)) = \beta \tag{4.5}$$

An example of such a function is $\bar{\beta}= 1/\beta$. Another example is the function β^* defined in section 2.2:

$$\beta^* = -\frac{1}{2} \ln (\tanh (\beta)) \tag{4.6}$$

All such functions, being symmetric about the line $\beta=\bar{\beta}$, have the property that $\bar{\beta}$ is large when β is small and vice-versa. Thus in the case of self-duality, it is the high and low temperature versions of the same theory which are related by the dual transformation.

4.2 Self-Duality for a Quantum Theory

For the case of the anisotropic two-dimensional Ising model considered in sec. 2.2, the dual transformation takes the following form[9]:

$$Z(\beta_\tau, \beta) = Z(\beta^*, \beta_\tau^*) \quad . \tag{4.7}$$

In terms of the transfer matrix[17], which was more easily expressed as a function of β and β_τ^* (see eq. (2.23)),

$$\hat{T}(\beta_\tau^*, \beta) \overset{d.t.}{\Longleftrightarrow} \hat{T}(\beta, \beta_\tau^*) \, . \tag{4.8}$$

That this is a consequence of (4.7) can be seen simply by noting that the parameters of $\hat{T}$ are obtained from those of the corresponding Z by taking the "*" of the first parameter and leaving the second alone. An arrow is written in this case to indicate that the $\hat{T}$ on the R.H.S. of (4.8) has the same form as the $\hat{T}$ on the L.H.S. They are equal if on the R.H.S. $\hat{T}$ is written with dual variables in place of the original variables (see 4.12). Since the coupling constant λ for the quantum Hamiltonian was defined as

$$\lambda = \beta / \beta_\tau^* \tag{4.9}$$

under the dual transformation it follows that

$$\hat{H}(\lambda) \Leftrightarrow \lambda H(1/\lambda) \, . \tag{4.10}$$

The transformation on the quantum Hamiltonian (as well as the transfer matrix) can be implemented by a canonical transformation on the operator variables. Recall

$$\hat{H}(\lambda, \hat{\sigma}) = -\sum_J \left(\hat{\sigma}_J^1 + \lambda \, \hat{\sigma}_J^3 \, \hat{\sigma}_{J+1}^3 \right) . \tag{4.11}$$

Let:

$$\hat{\mu}_J^1 = \hat{\sigma}_J^3 \hat{\sigma}_{J+1}^3 ,$$
$$\hat{\mu}_J^3 = \prod_{L=1}^{J} \hat{\sigma}_L^1 \quad . \tag{4.12}$$

The $\hat{\mu}$'s obey the same commutation relations as the $\hat{\sigma}$'s (See App. B).
From (4.12),

$$\hat{\sigma}_J^1 = \hat{\mu}_{J-1}^3 \hat{\mu}_J^3 \tag{4.13}$$

Therefore

$$\hat{H}(\lambda,\hat{\sigma}) = -\sum_J \left(\hat{\mu}_{J-1}^3 \hat{\mu}_J^3 + \lambda \hat{\mu}_J^1 \right) \tag{4.14}$$

Due to periodic boundary conditions, the indices on the first summand can be shifted by one, thus

$$\hat{H}(\lambda,\hat{\sigma}) = -\sum_J \left(\hat{\mu}_J^3 \hat{\mu}_{J+1}^3 + \lambda \hat{\mu}_J^1 \right) \tag{4.15}$$

$$\hat{H}(\lambda,\hat{\sigma}) = \lambda \hat{H}(1/\lambda, \hat{\mu}) \tag{4.16}$$

In the next section we shall show that (4.16) is the general statement of self-duality for a quantum Hamiltonian. In some ways, duality is a simpler matter in the quantum theory than it was in the classical statistical mechanical theory, since the transformation is simply a change of the operator variables. Translation of results from a theory to its dual version thus becomes a simple matter of substitution. That this was not true for the classical variables (s_i, u_i), is essentially due to the fact that s_i and u_i are eigenvalues of the operators $\hat{\sigma}^3$ and $\hat{\mu}^3$, and these operators cannot be simultaneously diagonalized.

4.3 Most General Self-Dual Hamiltonian

In order to determine the generality of the proof to be given in Chap. VII, it will be of use to consider the general form of a self-

dual Hamiltonian. In general one might only require

$$\tilde{H}(\lambda, \sigma) = K(\lambda)\, H(f(\lambda), \mu) \tag{4.17}$$

where σ , μ represent any set of operator-valued variables and their canonically equivalent dual variables (drop ^'s for readability). The duality requirement further implies

$$K(f(\lambda)) = 1/k(\lambda) \quad , \quad f(f(\lambda)) = \lambda \tag{4.18}$$

Equation (4.17) can be simplified by a change of variables. Let

$$\lambda = k^{-1}(\lambda') \tag{4.19}$$

Then

$$\tilde{H}(\lambda, \sigma) = \tilde{H}(K^{-1}(\lambda'), \sigma) = \lambda' H(f(k^{-1}(\lambda')), \mu) \tag{4.20}$$

Define

$$H'(\lambda', \sigma) = H(k^{-1}(\lambda'), \sigma) \tag{4.21}$$

$$f'(\lambda') = k(f(k^{-1}(\lambda'))) \tag{4.22}$$

Then (4.20) becomes

$$\tilde{H}'(\lambda', \sigma) = \lambda' H'(f'(\lambda'), \mu) \tag{4.23}$$

Also

$$f'(f'(\lambda)) = k(f(k^{-1}(k(f(k^{-1}(\lambda')))))) = k(f(f(k^{-1}(\lambda')))) = \lambda' \tag{4.24}$$

Taking the dual of (4.23), and then using (4.23), one obtains

$$H'(\lambda',\sigma) = \lambda'\tilde{H}'(f'(\lambda'),\mu) = \lambda' f'(\lambda') H'(\lambda',\sigma) \tag{4.25}$$

so

$$\lambda' f'(\lambda') = 1 \quad \Rightarrow \quad f'(\lambda') = 1/\lambda' \quad . \tag{4.26}$$

Therefore,

$$\tilde{H}'(\lambda',\sigma) = \lambda' H'(1/\lambda',\mu) \tag{4.27}$$

Since by changing parameters and redefining H one can put any self-dual Hamiltonian satisfying (4.17) into a form which satisfies (4.27), one can take the simpler (4.27) as a general definition of self-duality. Note therefore another simplification over the case of classical statistical mechanics. There is a single universal dual function $f(\lambda)=1/\lambda$.

Let H be made up of a set of operators and their duals, each multiplied by an arbitrary function of λ . Suppose a term in H has the form:

$$g(\lambda)\, A + h(\lambda)\, \tilde{A} \tag{4.28}$$

Then the term would satisfy (4.27) if

$$h(\lambda) = \lambda\, g(1/\lambda) \tag{4.29}$$

In order for H to be self-dual all sets of terms involving individual operators and their duals must alone be self-dual. So a general self-dual Hamiltonian would have the form:

$$H = \sum_i g_i(\lambda)\, A_i + \lambda\, g_i(1/\lambda)\, \tilde{A}_i \tag{4.30}$$

where the $g_i(\lambda)$ are arbitrary functions of λ (there would of course need to be some restrictions on the $g_i(\lambda)$ in order to insure H was bounded over some range of λ). The theorem of Chap VII applies to Hamiltonians of the form

$$H = k\mathcal{B} + \Gamma\tilde{\mathcal{B}} \tag{4.31}$$

where k,Γ are constants. Referring to (4.30) we see that (4.31) is the most general self-dual Hamiltonian which is linear in the coupling constant (take $g(\lambda)=k=\Gamma\lambda$).

V. Explicit Form of the Conserved Charges of the XZ Model

As was mentioned at the end of Chap. III, the higher charges for the XYZ model require a great deal of algebra to drive explicitly. Not only are there many derivatives to compute, but the task of constructing the cumulants to be subtracted from the charges G_N to obtain the local charges C_N is also formidable. I have computed the charges C_1 and C_2 in this manner, the latter requiring more than a few days work (see App. A for the result). If one considers the simpler XY model (J_z=0) or the equivalent XZ model (J_y=0) (which I choose to work with for reasons not relevant to this discussion), the form of the charges simplifies quite a lot. However, the procedure given by eq. (3.22) remains almost as difficult (setting J_y=0 helps only in the final stages).

I would like to describe a somewhat heuristic procedure which was used to determine the probable form of C_2 for the XZ model before the explicit calculation of App. A had been carried out.[22] This procedure does yield a conserved charge for C_2 and is rather easily extended to compute the form of the general charge C_N. Although it has not been proven that this is the same set of conserved charges as given by (3.22), the explicit calculation of App. A. does show that the procedure gives the correct result for C_2. It is also true that the first term in the charges to be calculated is taken from a term which occurs in the corresponding charge given by (3.22), so it is likely that the sets coincide. Even if they were not to, the procedure is still valid in that it generates an infinite set of conserved charges for the model. It is presented here since it provided the inspiration for the theorem of Chap. VII, in that it utilizes the self-duality of the model in the construction of conserved charges.

The XZ model, with Hamiltonian

$$H_{xz} = -\frac{1}{2}\sum_{J}\left(J_x\,\sigma_J^1\sigma_{J+1}^1 + J_z\,\sigma_J^3\sigma_{J+1}^3\right) \tag{5.1}$$

has an almost trivial dual transformation. If one lets

$$\mu_J^1 = \sigma_J^3\,,\quad \mu_J^3 = \sigma_J^1 \tag{5.2}$$

then

$$H_{XZ} = -\frac{1}{2}\sum_J \left(J_x\, \mu_J^3 \mu_{J+1}^3 + J_z\, \mu_J^1 \mu_{J+1}^1\right) \tag{5.3}$$

which is the same as the original Hamiltonian but with the coupling J_x/J_z inverted. Thus the model is self-dual.

In calculating C_2,

$$C_2 = G_2 - 3\, G_0 G_1 + 2\, G_0^3 \tag{5.4}$$

one must first calculate G_2. The term which differs in structure from those encountered previously is that in which derivatives hit three adjacent sites (adjacent values of J) in T (3.5). It is reasonable to assume that a term of this sort is the most non-local term which will survive when the local charge C_2 is calculated by (5.4). Also, it can be shown fairly easily that the symmetric part of the mentioned term is subtracted off by the lower cumulants in (5.4) as was the case for C_1. Thus we are left with

$$\begin{aligned} C_2 &= \sum_J \sum_{i,j,k=1}^{4} p_i' p_j' p_k'\, \sigma_J^i [\sigma_{J+1}^i, \sigma_{J+1}^j][\sigma_{J+2}^j, \sigma_{J+2}^k]\, \sigma_{J+3}^k + \text{other terms} \\ &= \sum_J \sum_{i,j,k=1}^{3} p_i' p_j' p_k'\, \epsilon_{ija}\, \epsilon_{jkb}\, \sigma_J^i \sigma_{J+1}^a \sigma_{J+2}^b \sigma_{J+3}^k + \text{other terms.} \end{aligned} \tag{5.5}$$

If one sets $p_2' \propto J_y = 0$, and defines B_2 as the first term of (5.5) in this limit one obtains

$$B_2 = \sum_J \left(p_1'^2 p_3'\, \sigma_J^1 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^1 + p_1' p_3'^2\, \sigma_J^3 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^3 \right). \tag{5.6}$$

The first thing to test is whether B_2 itself is conserved. In computing the commutator with H_{XZ} (5.1) one notices that it is possible for the commutator of the first term of H_{XZ} with the second term of B_2 to cancel with the opposite cross term, since they have the same co-

efficients, but the direct term commutators could not cancel since one has a coefficient of $p_1'^3 p_3'$ and the other $p_1' p_3'^3$. Indeed this is what happens when the commutator is calculated:

$$[H_{xz}, B_2] = -i\left(p_1'^3 p_3' - p_1' p_3'^3\right)\sum_J \left(\sigma_J^1 \sigma_{J+1}^2 \sigma_{J+2}^3 + \sigma_J^3 \sigma_{J+1}^2 \sigma_{J+2}^1\right). \tag{5.7}$$

It is very interesting, however, that the commutator is zero at the self-dual point, $J_x = J_z$ ($p_1' = p_3'$). This suggests that another quantity which is also conserved at the self-dual point, but not away from that point should be added to B_2 to obtain a conserved charge. An obvious candidate would be the dual of a lower conserved charge. Since C_1 is self-dual, the only other choice is C_0, i.e. the dual of the Hamiltonian itself. The relevant commutator is

$$[H_{xz}, \tilde{H}_{xz}] = -\frac{i}{2}\left(p_1'^2 - p_3'^2\right)\sum_J \left(\sigma_J^1 \sigma_{J+1}^2 \sigma_{J+2}^3 + \sigma_J^3 \sigma_{J+1}^2 \sigma_{J+2}^1\right) \tag{5.8}$$

$$(J_x = -p_1',\ J_y = -p_2',\ J_z = -p_3')$$

Thus one sees that this is indeed the correct choice and the combination

$$\bar{C}_2 = B_2 - 2 p_1' p_3' \tilde{H}_{xz} \tag{5.9}$$

is conserved for all p_1', p_3'. Specifically

$$\bar{C}_2 \propto \sum_J \Big(p_1' \sigma_J^1 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^1 + p_3' \sigma_J^3 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^3 + p_3' \sigma_J^1 \sigma_{J+1}^1 + p_1' \sigma_J^3 \sigma_{J+1}^3\Big) \tag{5.10}$$

If we take the previously computed expression for C_1 and specialize to the case $J_y \propto p_2' = 0$, calling the result $\bar{C}_1$ we obtain

$$\bar{C}_1 = -\frac{i}{2} \rho_1' \rho_3' \sum_J \left(\sigma_J^1 \sigma_{J+1}^2 \sigma_{J+2}^3 - \sigma_J^3 \sigma_{J+1}^2 \sigma_{J+2}^1 \right). \tag{5.11}$$

Note that $\bar{C}_2$ depends upon the coupling constants in an important way, whereas $\bar{C}_1$ depends only trivially upon them. Also $\bar{C}_1$ is a self-dual operator, whereas $\bar{C}_2$ is only self-dual in the same sense that the Hamiltonian is, i.e. its dual is the same operator but with the couplings interchanged.

It is now clear how to proceed for the higher charges. Let

$$\begin{aligned} B_3 &= \sum_J \sum_{i,j,k,\ell=1}^{3} \rho_i' \rho_j' \rho_k' \rho_\ell' \, \epsilon_{ija} \epsilon_{jkb} \epsilon_{k\ell c} \, \sigma_J^i \sigma_{J+1}^a \sigma_{J+2}^b \sigma_{J+3}^c \sigma_{J+4}^\ell \\ &= \rho_1'^2 \rho_3'^2 \sum_J \left(\sigma_J^1 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^2 \sigma_{J+4}^3 - \sigma_J^3 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^2 \sigma_{J+4}^1 \right). \end{aligned} \tag{5.12}$$

This already commutes with the Hamiltonian so it can be identified as $\bar{C}_3$. Note its similarly to $\bar{C}_1$. It is clear that all of the odd numbered charges will have this form, each successive charge having the string of σ^2's within its terms lengthened by two sites. Going on,

$$B_4 = \sum_J \left(\rho_1'^3 \rho_3'^2 \sigma_J^1 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^2 \sigma_{J+4}^2 \sigma_{J+5}^1 + \rho_1'^2 \rho_3'^3 \sigma_J^3 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^2 \sigma_{J+4}^2 \sigma_{J+5}^3 \right), \tag{5.13}$$

$$\left[H_{xz}, B_4 \right] = -i \left(\rho_1'^4 \rho_3'^2 - \rho_1'^2 \rho_3'^4 \right) \sum_J \left(\sigma_J^1 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^2 \sigma_{J+4}^3 + \sigma_J^3 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^2 \sigma_{J+4}^1 \right). \tag{5.14}$$

Thus B_4, like B_2, is conserved at the self-dual point. Try

$$[H_{xz}, \tilde{B}_2] = i(P_1'^3 P_3' - P_1' P_3'^3) \sum_J (\sigma^1_J \sigma^2_{J+1} \sigma^2_{J+2} \sigma^2_{J+3} \sigma^3_{J+4} + \sigma^3_J \sigma^2_{J+1} \sigma^2_{J+2} \sigma^2_{J+3} \sigma^1_{J+4}). \tag{5.15}$$

Thus

$$[H_{xz}, B_4 + P_1' P_3' \tilde{B}_2] = 0 \tag{5.16}$$

$$\bar{C}_4 = \sum_J \Big(P_1' \sigma^1_J \sigma^2_{J+1} \sigma^2_{J+2} \sigma^2_{J+3} \sigma^2_{J+4} \sigma^1_{J+5} + P_3' \sigma^3_J \sigma^2_{J+1} \sigma^2_{J+2} \sigma^2_{J+3} \sigma^2_{J+4} \sigma^3_{J+5} + P_3' \sigma^1_J \sigma^2_{J+1} \sigma^2_{J+2} \sigma^1_{J+3} + P_1' \sigma^3_J \sigma^2_{J+1} \sigma^2_{J+2} \sigma^3_{J+3}\Big). \tag{5.17}$$

One can generalize these forms to the N^{th} charge:

$$\bar{C}_{2k-1} \propto \sum_J (\sigma^3_J \sigma^2_{J+1} \sigma^2_{J+2} \cdots \sigma^2_{J+2k-1} \sigma^1_{J+2k} - \sigma^1_J \sigma^2_{J+1} \sigma^2_{J+2} \cdots \sigma^2_{J+2k-1} \sigma^3_{J+2k}) \tag{5.18}$$

$$\bar{C}_{2k} \propto \sum_J \big(P_1' \sigma^1_J \sigma^2_{J+1} \sigma^2_{J+2} \cdots \sigma^2_{J+2k} \sigma^1_{J+2k+1} + P_3' \sigma^3_J \sigma^2_{J+1} \sigma^2_{J+2} \cdots \sigma^2_{J+2k} \sigma^3_{J+2k+1} + P_3' \sigma^1_J \sigma^2_{J+1} \sigma^2_{J+2} \cdots \sigma^2_{J+2k-2} \sigma^1_{J+2k-1} + P_1' \sigma^3_J \sigma^2_{J+1} \sigma^2_{J+2} \cdots \sigma^2_{J+2k-2} \sigma^3_{J+2k-1}\big). \tag{5.19}$$

It can be shown that all of these charges commute with each other as well as with the Hamiltonian. This is true for the XYZ charges also as can be seen by differentiating (3.11) with respect to V and V' an arbitrary number of times and making use of (3.17). The existence of an infinite commuting set of conserved charges is a hallmark of exact integrability, as will be seen in the next chapter.

The explicit forms of the charges (5.18 and 5.19) give a clue as to why this model and others like it possess an infinite set of local conserved charges. The basic idea is that H_{XZ} commutes with a string of σ^2 operators no matter how long, except at the endpoints. If the end points of several strings of this sort can be chosen so that H_{XZ}

commutes with the combination then one has not found just one conserved charge but an infinite number since the original string length was arbitrary. This is very much a two-dimensional phenomena (one-spatial dimension). One reason is that one-dimensional strings are the only geometrical objects whose boundary (two points) remains the same size as the string grows (compare a circle or sphere). Thus any attempt to generalize the above argument to charges whose currents have non-unit support on higher geometrical objects is probably doomed. If one retains a string form for the currents in the higher dimensional case then another problem arises. This is that the commutators of the Hamiltonian with a string can have shapes (i.e. regions of non-unit support) other than a simple linear string making it impossible to cancel against the commutator of the Hamiltonian with another string in a different place, as is the construction used in the charges for H_{XZ}. A more detailed analysis along these lines is given in ref. 22.

The conserved charges for the XYZ model clearly have a more complicated form. The result of App. A shows that the coefficients of terms in the currents become very complicated polynominals in the couplings. This makes even checking the conservation of a charge a difficult task. It would be very interesting to know the explicit form of higher charges for the XYZ model, and, in particular whether they also contain a simple repeating pattern which commutes with the Hamiltonian and can be extended indefinitely. A computer algebra program could probably generate enough charges to answer this question.

Once a pattern is discerned, one can try to find an algebra obeyed by the charges which will generate the higher charges from components of the lower ones instead of from the generating functional. For the XZ model it is possible to obtain all of the charges from a commutator algebra involving the operators in the Hamiltonian. This algebra is given in Chap. VII (7.1-3) where it is generalized to the case of any self-dual Hamiltonian. There it is proven that the resulting infinite set of charges will always be conserved, provided the first charge in the set is conserved.

VI. Conserved Charges from the Quantum Inverse Scattering Method

In this chapter I would like to briefly illustrate some of the features of the quantum inverse scattering method (QISM).[23] This method has proven useful in the diagonalization of certain 1+1 dimensional interacting quantum field theories, such as the quantum sine-Gordon equation and the XYZ model. In particular, I would like to show how an infinite set of conserved charges arises naturally out of the method, using as an example the XYZ model. Of course this model was originally solved by Baxter without the use of this method,[20] however there are many similarities between the method of Baxter and the QISM as has been pointed out by Takhtadzhan and Faddeev.[24] Starting with the XYZ model Hamiltonian for a chain of length 2N with periodic boundary conditions,

$$H = \frac{1}{2} \sum_{k=-N}^{N-1} \sum_{i=1}^{3} J_i \, \sigma_k^i \, \sigma_{k+1}^i \, , \qquad (6.1)$$

one can find equations of motion:

$$\dot{\sigma}_k^i = i[H, \sigma_k^i] = \sum_{\ell, m=1}^{3} \epsilon_{i\ell m} J_\ell \left(\sigma_{k+1}^\ell + \sigma_{k-1}^\ell \right) \sigma_k^m \, . \qquad (6.2)$$

In the QISM one attempts to find a local pair of matrices $L_K(\lambda), M_K(\lambda)$, such that

$$-i \dot{L}_k(\lambda) = M_{k+1}(\lambda) L_k(\lambda) - L_k(\lambda) M_k(\lambda) = [H, L_k(\lambda)] \qquad (6.3)$$

for all K, λ and so that (6.3) is true if and only if the equations of motion (6.2) are satisfied (This will generally be true if L_K is a simple function of the σ's). There is no systematic procedure for finding such matrices. One generally chooses an ansatz for the general form of the matrices and uses (6.3) to attempt to solve for the correct coefficient functions. There do exist heuristic procedures for choosing a promising ansatz (see e.g. ref. 25). An infinite set of conserved charges arises from the quantity

$$\mathcal{L}(\lambda) = \prod_{K=N}^{K=-N+1} L_K(\lambda) \tag{6.4}$$

which is known as the monodromy matrix. From (6.3) one can see that $\mathrm{tr}\,\mathcal{L}(\lambda)$ commutes with H. Thus

$$D_N = \frac{\partial^{N+1}}{\partial\lambda^{N+1}} \mathrm{tr}\,\mathcal{L}(\lambda)\Big|_{\lambda=\lambda_0} \qquad N = 0, 1, 2, \cdots \tag{6.5}$$

form an infinite set of conserved charges (λ_0 may be chosen arbitrarily in order to produce a convenient set). Of course the λ dependence must be non-trivial in order that the charges be independent of one another, and in fact the method will not work unless this is so. Also, for a finite chain the number of independent charges is finite - one per degree of freedom. Only as $N \to \infty$ is there an infinite set. Generally the form of $L_K(\lambda)$ can be chosen so that

$$D_0 = H. \tag{6.6}$$

The main goal of the QISM is to find the eigenvalues and eigenvectors of the Hamiltonian. This contrasts somewhat with the goal of the classical inverse method which is to solve the initial value problem. The method diagonalizes H by first diagonalizing $\mathcal{L}(\lambda)$ with which it shares a common set of eigenvectors. If (6.6) holds then the eigenvalues can be obtained from those of $\mathcal{L}(\lambda)$ by expanding in λ. The diagonalization of $\mathcal{L}(\lambda)$ is accomplished by solving an associated linear problem given by

$$\Psi_{K+1} = L_K(\lambda)\Psi_K \tag{6.7}$$

the details of which can be found in ref. 24.

In the case of the XYZ model, which I am discussing, one correct choice of $L_K(\lambda)$, $M_K(\lambda)$ has the feature that the trace of the monodromy

matrix is exactly equal to the transfer matrix of the 8-vertex model (3.5). Thus the charges derived before from the transfer matrix are the same ones which result from the complete quantum integrability of the system.

In simple terms, the QISM can be thought of as basically a transformation to action angle variables Q_i, P_i, such that H is a function of the P_i only. Then it immediately results that

$$\dot{P}_i = 0 \; . \tag{6.8}$$

An infinite set of conserved charges is therefore seen to be an integral feature of the quantum inverse method.

VII Construction of an Infinite Set of Conserved Charges for Self-Dual Theories

In this section I construct an infinite set of charges for a specific class of self-dual theories, namely those whose Hamiltonians can be written in the form

$$H = \kappa B + \Gamma \tilde{B} \tag{7.1}$$

where K and Γ are coupling constants and B is some operator.[26] The form of the dual transformation need not be specified beyond that it is a linear operation which changes B to $\tilde{B}$ and $\tilde{B}$ to B. Only one additional condition is needed to guarantee the existence of an infinite set of charges. The charges are given by

$$Q_{2n} \equiv K\,(W_{2n} - \tilde{W}_{2n-2}) + \Gamma\,(\tilde{W}_{2n} - W_{2n-2}) \tag{7.2}$$

$$n = 1, 2, 3, \ldots$$

where

$$W_{2n+2} \equiv -\frac{1}{8}\,[B,[\tilde{B}, W_{2n}]] - \tilde{W}_{2n} \tag{7.3}$$

and $W_0 \equiv B$, n=1,2,3,.... . The charges have been labelled with positive even integers in order to match as closely as possible the notation of reference 22. In order for Q_{2N}, to be a conserved charge,

$$[Q_{2N}, H] = 0 \tag{7.4}$$

it is sufficient to show that

$$[B, W_{2n}] = [B, \tilde{W}_{2n-2}] \tag{7.5a}$$

$$[\tilde{B}, W_{2n}] = -[B, \tilde{W}_{2n}] \tag{7.5b}$$

for $1 \leq n \leq N$. I shall prove (7.5) inductively for all N. In order to perform the induction in N it is easier to consider (7.5b) with $n \to n-1$

as the natural partner to (7.5a), i.e. I shall prove the following set inductively for all N:

$$[B, W_{2n}] = [B, \widetilde{W}_{2n-2}] \tag{7.6a}$$

$$[\widetilde{B}, W_{2n-2}] = -[B, \widetilde{W}_{2n-2}] \tag{7.6b}$$

for $1 \leq n \leq N$. First, consider n=1.

$$[B, W_2] = -\tfrac{1}{8}[B, [B, [\widetilde{B}, B]]] - [B, \widetilde{B}]. \tag{7.7}$$

One sees that (7.6a) will be true only if

$$[B, [B, [B, \widetilde{B}]]] = 16\,[B, \widetilde{B}]. \tag{7.8}$$

This shall have to be assumed as an auxiliary condition to (7.1). Equation (7.6b) is seen to be trivially satisfied for n=1. Going on to n=2, observe that

$$\begin{aligned}
[B, W_4] &= -\tfrac{1}{8}[B, [B, [\widetilde{B}, W_2]]] - [B, \widetilde{W}_2] \\
&= \tfrac{1}{64}[B, [B, [\widetilde{B}\, [B, [\widetilde{B}, B]]]]] - [B, \widetilde{W}_2] \\
&= \tfrac{1}{64}[B, [[B, \widetilde{B}], [B, [\widetilde{B}, B]]]] + \tfrac{1}{64}[B, [\widetilde{B}, [B, [B, [\widetilde{B}, B]]]]] - [B, \widetilde{W}_2] \\
&= \tfrac{1}{64}[[B, \widetilde{B}], [B, [B, [\widetilde{B}, B]]]] + \tfrac{1}{64}[B, [\widetilde{B}, [B, [B, [\widetilde{B}, B]]]]] - [B, \widetilde{W}_2]
\end{aligned} \tag{7.9}$$

Using (7.8), one finds

$$[B, W_4] = \tfrac{1}{4}[B, [\widetilde{B}, [\widetilde{B}, B]]] - [B, \widetilde{W}_2] = [B, \widetilde{W}_2] \tag{7.10}$$

so (7.6a) is valid for n=2.

$$[\tilde{B}, W_2] = -\frac{1}{8}[\tilde{B},[B,[\tilde{B},B]]] = -\frac{1}{8}[B,[\tilde{B},[\tilde{B},B]]] = -[B,\tilde{W}_2] \quad (7.11)$$

so (7.6b) is also valid for n=2.

It will be useful to consider an alternative formula for W_{2n+2}, valid when (7.6) is valid for a given n. From the definition (7.3),

$$\begin{aligned} W_{2n+2} &= -\frac{1}{8}[B,[\tilde{B}, W_{2n}]] - \tilde{W}_{2n} \\ &= -\frac{1}{8}[[B,\tilde{B}], W_{2n}] - \frac{1}{8}[\tilde{B},[B,W_{2n}]] - \tilde{W}_{2n} \, . \end{aligned} \quad (7.12)$$

Applying (7.6) one obtains

$$W_{2n+2} = -\frac{1}{8}[[B,\tilde{B}], W_{2n}] - \frac{1}{8}[\tilde{B},[B,\tilde{W}_{2n-2}]] - \tilde{W}_{2n} \quad (7.13)$$

$$W_{2n+2} = -\frac{1}{8}[[B,\tilde{B}], W_{2n}] + W_{2n-2} \, . \quad (7.14)$$

Equation (7.14) can be extended to n=0 by defining

$$W_{-2} \equiv -\tilde{W}_0 \, . \quad (7.15)$$

It can be further extended to n<0 by defining

$$W_{-2r} \equiv -\tilde{W}_{2r-2} \quad (7.16)$$

for r=1,2,3, ... , as can be seen by taking the dual of (7.14). Thus if (7.6) is valid in the range $1 \leq n \leq N$, (7.14) will be valid in the range $-N-1 \leq n \leq N$. Instead of proving only equations (7.6), it will be conceptually easier to prove a more general equation (7.17). Furthermore, proof of (7.17) results in not only $[H, Q_{2n}]=0$ but also $[Q_{2n}, Q_{2m}]=0$, i.e. the conserved charges all commute, a hallmark of

exact integrability. I will prove:

$$[\widetilde{W}_{2\ell}, W_{2n-2\ell-2}] = [\widetilde{W}_{2\ell-2}, W_{2n-2\ell}] \tag{7.17}$$

for all $n \geq 0$, $\ell \geq 0$. The equation then follows for all ℓ, since if one lets $\ell = n+p$ in (7.17), $p=1,2, \ldots$, and takes the dual of the equation one obtains

$$[W_{2n+2p}, \widetilde{W}_{-2p-2}] = [W_{2n+2p-2}, \widetilde{W}_{-2p}] \qquad p = 1,2,3,\cdots \tag{7.18}$$

which is (7.17) with $\ell=-p$. It then also follows for $n<0$ since with use of (7.16), (7.17) is equivalent to

$$[W_{-2\ell-2}, \widetilde{W}_{-2n+2\ell}] = [W_{-2\ell}, \widetilde{W}_{-2n+2\ell-2}]. \tag{7.19}$$

Letting $n'=-n, \ell'=-\ell$, and taking the dual of (7.19), one obtains

$$[\widetilde{W}_{2\ell'}, W_{2n'-2\ell'-2}] = [\widetilde{W}_{2\ell'-2}, W_{2n'-2\ell'}] \tag{7.20}$$

which is valid for $n' \leq 0$, $-\infty < \ell' < \infty$. Thus if (7.17) is proven for $n \geq 0$, $\ell \geq 0$ it will also be true for all integral n and ℓ. If one sets $\ell=0$ in (7.17), one obtains

$$[\widetilde{B}, W_{2n-2}] = -[B, W_{2n}] \tag{7.21}$$

which is (7.6a) plus (7.6b). When (7.17) is added together for $\ell=1$, $\ell=2$, ... , $\ell=n-1$, the result is

$$[\widetilde{W}_{2n-2}, B] = [\widetilde{B}, W_{2n-2}] \tag{7.22}$$

which is (7.6b). Thus it is sufficient to prove the more general equation (7.17).

For $n=0$, (7.17) is trivially valid for $-\infty<\ell<\infty$. The case ($n=1$; $\ell=0,1$) follows from (7.8) as shown earlier. I also have proven the

case ($n=2, \ell=0,1,2$) explicitly: (7.10), (7.11). I now prove (7.17) by induction. Assume (7.17) to be true for

$$\begin{array}{lll}
n=N & , & 0\le \ell \le N \\
n=N-1 & , & 0\le \ell \le N-1 \\
n=N-2 & , & 0\le \ell \le N-1 \\
n=N-3 & , & 0\le \ell \le N-2 \\
n=N-4 & , & 0\le \ell \le N-2 \\
\vdots & & \vdots \\
n=N-k & , & 0\le \ell \le N-\frac{1}{2}k=\frac{1}{2}(N+n) \\
\vdots & & \vdots \\
n=2 & & 0\le \ell \le \frac{1}{2}N+1 \\
n=1 & & 0\le \ell \le \frac{1}{2}N+\frac{1}{2} \\
n=0 & & 0\le \ell \le \frac{1}{2}N
\end{array} \tag{7.23}$$

(n and ℓ are always integral).

Eq. (7.17) has already been shown for $N=2$. This will serve as the base level for the induction. I now must prove (7.23) for $N \to N+1$, i.e. I must show (7.17) to be true when

$$
\begin{array}{ll}
n = N+1 , & 0 \le \ell \le N+1 \\
n = N , & 0 \le \ell \le N \\
n = N-1 , & 0 \le \ell \le N \\
n = N-2 , & 0 \le \ell \le N-1 \\
n = N-3 , & 0 \le \ell \le N-1 \\
n = N-4 , & 0 \le \ell \le N-2 \\
\vdots & \vdots \\
n = N-k , & 0 \le \ell \le N - \tfrac{1}{2}k + \tfrac{1}{2} = \tfrac{1}{2}(N+n+1) \\
\vdots & \vdots \\
n = 2 & 0 \le \ell \le \tfrac{1}{2}N + \tfrac{3}{2} \\
n = 1 & 0 \le \ell \le \tfrac{1}{2}N + 1 \\
n = 0 & 0 \le \ell \le \tfrac{1}{2}N + \tfrac{1}{2} .
\end{array}
\tag{7.24}
$$

Note that in addition to adding a new level to the induction (the first equation of (7.24), n=N+1) one must also raise the limit on ℓ on half of the previous levels. The proof will consist of three parts. First I will prove (7.17) for n=N+1, $1 \le \ell \le N$. Then I will extend this to $\ell=0$ and $\ell=N+1$. Finally I will raise the limit on ℓ for the previous levels, n<N.

Part I. Commute $-\frac{1}{8}[B,\tilde{B}]$ with (7.17) for n=N.

$$-\tfrac{1}{8}[[B,\tilde{B}],[\tilde{W}_{2\ell},W_{2N-2\ell-2}]] = -\tfrac{1}{8}[[B,\tilde{B}],[\tilde{W}_{2\ell-2},W_{2N-2\ell}]] . \tag{7.25}$$

Then,

$$-\tfrac{1}{8}[[[B,\tilde{B}],\tilde{W}_{2\ell}],W_{2N-2\ell-2}] - \tfrac{1}{8}[\tilde{W}_{2\ell},[[B,\tilde{B}],W_{2N-2\ell-2}]]$$

$$= -\tfrac{1}{8}[[[B,\tilde{B}],\tilde{W}_{2\ell-2}],W_{2N-2\ell}] - \tfrac{1}{8}[\tilde{W}_{2\ell-2},[[B,\tilde{B}],W_{2N-2\ell}]] . \tag{7.26}$$

For $0 \leq \ell \leq N$, (7.14) allows one to write

$$-[\widetilde{W}_{2\ell+2}, W_{2N-2\ell-2}] + [\widetilde{W}_{2\ell-2}, W_{2N-2\ell-2}] + [\widetilde{W}_{2\ell}, W_{2N-2\ell}] - [\widetilde{W}_{2\ell}, W_{2N-2\ell-4}]$$
$$= -[\widetilde{W}_{2\ell}, W_{2N-2\ell}] + [\widetilde{W}_{2\ell-4}, W_{2N-2\ell}] + [\widetilde{W}_{2\ell-2}, W_{2N-2\ell+2}] - [\widetilde{W}_{2\ell-2}, W_{2N-2\ell-2}]. \tag{7.27}$$

With the assumption (7.23) for n=N-1, Equation (7.27) gives, for $1 \leq \ell \leq N-1$,

$$2[\widetilde{W}_{2\ell}, W_{2N-2\ell}] = [\widetilde{W}_{2\ell+2}, W_{2N-2\ell-2}] + [\widetilde{W}_{2\ell-2}, W_{2N-2\ell+2}] . \tag{7.28}$$

Each of these equations is the sum of two adjacent equations in the desired set ((7.17) with n=N+1). Equation (7.17) with n=N+1 and $1 \leq \ell \leq N$ is a system of N equations in N+1 unknowns (it states that all of the considered commutators are equal). Equation (7.28) is a set of only N-1 equations in the same N+1 unknowns. The former set implies the latter set. To prove (7.17) (with n=N+1, $1 \leq \ell \leq N$) from (7.28) therefore, one more condition is needed. There are two cases, depending upon whether N is odd or even.

Case 1: N is odd, $N \geq 3$. I will now prove the additional equation $[\widetilde{W}_{N+1}, W_{N-1}] = [\widetilde{B}, W_{2N}]$. From the assumption (7.23) with n=N

$$[\widetilde{B}, W_{2N-2}] = [\widetilde{W}_{N-1}, W_{N-1}] . \tag{7.29}$$

Then

$$-\tfrac{1}{8}[\widetilde{B},[B,[\widetilde{B},W_{2N-2}]]] = -\tfrac{1}{8}[\widetilde{B},[B,[\widetilde{W}_{N-1},W_{N-1}]]] \tag{7.30}$$

$$[\widetilde{B},W_{2N}] + [\widetilde{B},\widetilde{W}_{2N-2}] = -\tfrac{1}{8}[\widetilde{B},[[B,\widetilde{W}_{N-1}],W_{N-1}]] - \tfrac{1}{8}[\widetilde{B},[\widetilde{W}_{N-1},[B,W_{N-1}]]] \tag{7.31}$$

$$[\tilde{B}, W_{2N}] + [\tilde{B}, \tilde{W}_{2N-2}] = [\tilde{W}_{N+1}, W_{N-1}] - \frac{1}{8}[[B, \tilde{W}_{N-1}], [\tilde{B}, W_{N-1}]]$$
$$- \frac{1}{8}[[\tilde{B}, W_{N-3}], [B, \tilde{W}_{N-3}]] + [\tilde{W}_{N-1}, W_{N-3}] \tag{7.32}$$

where I have made use of the Jacobi identity, definition (7.3), and the assumption (7.23) with $n= \frac{1}{2}(N-1)$. Using this same assumption again I obtain

$$[\tilde{B}, W_{2N}] + [\tilde{B}, \tilde{W}_{2N-2}] = [\tilde{W}_{N+1}, W_{N-1}] + [\tilde{W}_{N-1}, W_{N-3}]. \tag{7.33}$$

Finally using the assumption (7.23) with $n=N-1$, $\ell= \frac{1}{2}(N-1)$, one gets

$$[\tilde{B}, W_{2N}] = [\tilde{W}_{N+1}, W_{N-1}] \tag{7.34}$$

which is an equation derivable from (7.17) with $n=N+1$, $1\leq\ell\leq N$ and independent of (7.28). Together (7.28) and (7.34) prove (7.17) with $n=N+1$, $1\leq\ell\leq N$, N odd. The crucial step in this proof is the cancellation of the second term on the right hand side of (7.32). If I had chosen an arbitrary equation from (7.17) as the starting point, instead of (7.29), then the corresponding term would not have cancelled.

Case 2: N is even, $N\geq 2$. For this case, the extra condition derived is $[\tilde{B}, W_{2N}] = -[B, \tilde{W}_{2N}]$. To prove this I first show an intermediate result, based on assumptions (7.23):

$$\frac{1}{8}[[B, \tilde{W}_{2p}], [\tilde{B}, W_{2m-2p-2}]] = (p+1)\left([B, \tilde{W}_{2m}] + [\tilde{B}, W_{2m}]\right) \tag{7.35}$$

for integral p, $0\leq 2p\leq m-1$ and $1\leq m\leq N$. First, (7.35) is true for $p=0$.

$$\frac{1}{8}[[B, \tilde{B}], [\tilde{B}, W_{2m-2}]] = -\frac{1}{8}[B, [\tilde{B}, [B, \tilde{W}_{2m-2}]]] - \frac{1}{8}[\tilde{B}, [B, [\tilde{B}, W_{2m-2}]]]$$
$$= [B, \tilde{W}_{2m}] + [\tilde{B}, W_{2m}] \tag{7.36}$$

I have used the fact that $[B, W_{2m-2}]$ is anti-self-dual (a.s.d.) as a consequence of the assumption (7.23) with n=m-1. Since (7.35) is true for p=0 and all m, $1 \le m \le N$, the desired result is true for m=1 and m=2 (for these cases p is restricted to the value zero). I shall now prove (7.35) inductively in m. Assume (7.35) for $1 \le m \le K-1$ for some K, $3 \le K \le N$ and show it is true for m=K. Assume p>0, since the result for p=0 has already been shown, and also that the restriction on p following (7.35) holds.

$$
\begin{aligned}
\tfrac{1}{8}[[B,\tilde W_{2p}],[\tilde B, W_{2K-2p-2}]] &= -\tfrac{1}{64}[[B,[\tilde B,[B,\tilde W_{2p-2}]]],[\tilde B, W_{2K-2p-2}]] \\
&\quad -\tfrac{1}{8}[[B,\tilde W_{2p-4}],[\tilde B, W_{2K-2p-2}]] \\
&= \tfrac{1}{64}[[\tilde B,[B,\tilde W_{2p-2}]],[B,[\tilde B, W_{2K-2p-2}]]] \\
&\quad -\tfrac{1}{64}[B,[[\tilde B,[B,\tilde W_{2p-2}]],[\tilde B, W_{2K-2p-2}]]] \\
&\quad -\tfrac{1}{8}[[B,\tilde W_{2p-4}],[\tilde B, W_{2K-2p-2}]]
\end{aligned}
\tag{7.37}
$$

If $p \ge 2$, then one can apply (7.35) with m=K-2, to the last term in (7.37). It can then be shown to be zero by the appropriate assumption in (7.23) (specifically in the form (7.6b) with n=K-1). If p=1 this term is zero, since $\tilde W_{-2} = -B$. Therefore,

$$
\begin{aligned}
\tfrac{1}{8}[[B,\tilde W_{2p}],[\tilde B, W_{2K-2p-2}]] &= \tfrac{1}{8}[[B,\tilde W_{2p-2}],[\tilde B, W_{2K-2p}]] \\
&\quad +\tfrac{1}{8}[[B,\tilde W_{2p-2}],[\tilde B, W_{2K-2p-4}]] + \tfrac{1}{64}[\tilde B,[[B,\tilde W_{2p-2}],[B,[\tilde B, W_{2K-2p-2}]]]] \\
&\quad -\tfrac{1}{64}[B,[\tilde B,[[B,\tilde W_{2p-2}],[\tilde B, W_{2K-2p-2}]]]] \\
&\quad +\tfrac{1}{64}[B,[[\tilde B, W_{2p-2}],[\tilde B,[B,\tilde W_{2K-2p-2}]]]] .
\end{aligned}
\tag{7.38}
$$

One can apply (7.35) with m=K-2 to the second term, since $2p-2 \le K-3$, the condition for applicability, follows from the restriction on p for m=K: $2p \le K-1$. The second term is then zero by (7.23). Similarly, the fourth term is zero by (7.35) with m=K-1 and (7.23) with n=K. Also note that the third and fifth terms are the duals of each other. So

$$\frac{1}{8}[[B,\tilde{W}_{2p}],[\tilde{B},W_{2K-2p-2}]] - \frac{1}{8}[[B,\tilde{W}_{2p-2}],[\tilde{B},W_{2K-2p}]]$$
$$= -\frac{1}{8}[\tilde{B},[[B,\tilde{W}_{2p-2}],W_{2K-2p}]] - \frac{1}{8}[\tilde{B},[[B,\tilde{W}_{2p-2}],\tilde{W}_{2K-2p-2}] + d. \tag{7.39}$$

where d. indicates the dual of the expression. Consider the right hand side (RHS) of (7.39):

$$RHS = \frac{1}{8}[\tilde{B},[\tilde{W}_{2p-2},[B,W_{2K-2p}]]] - \frac{1}{8}[\tilde{B},[B,[\tilde{W}_{2p-2},W_{2K-2p}]]]$$
$$-\frac{1}{8}[[B,\tilde{W}_{2p-2}],[\tilde{B},W_{2K-2p-4}]] - \frac{1}{8}[[\tilde{B},[B,\tilde{W}_{2p-2}]],\tilde{W}_{2K-2p-2}] + d. \tag{7.40}$$

As before the third term is zero.

$$RHS = \frac{1}{8}[[B,\tilde{W}_{2p-4}],[\tilde{B},W_{2K-2p-2}]] + \frac{1}{8}[\tilde{W}_{2p-2},[\tilde{B},[B,\tilde{W}_{2K-2p-2}]]]$$
$$-\frac{1}{8}[\tilde{B},[B,[\tilde{B},W_{2K-2}]]] + [\tilde{W}_{2p},\tilde{W}_{2K-2p-2}] + [W_{2p-2},\tilde{W}_{2K-2p-2}] + d. \tag{7.41}$$

Again, the first term vanishes. Consider the fourth term:

$$[\tilde{W}_{2p},\tilde{W}_{2K-2p-2}] = [\tilde{W}_{2K-2p-2},W_{-2p-2}]. \tag{7.42}$$

Let $\ell=K-p-1$, $n=K-2p-1$. Then

$$[\tilde{W}_{2p},\tilde{W}_{2K-2p-2}] = [\tilde{W}_{2\ell},W_{2n-2\ell-2}]. \tag{7.43}$$

If one eliminates p, then

$$\ell = \tfrac{1}{2}(K+n-1) \le \tfrac{1}{2}(N+n-1). \tag{7.44}$$

Since n<N (7.43) is therefore covered by the assumption (7.23) which implies that it is a.s.d.

$$[\widetilde{W}_{2p}, \widetilde{W}_{2K-2p-2}] + \mathrm{d.} = 0 \tag{7.45}$$

The fifth term of (7.41) is also a.s.d. by (7.23) with n=K-1. So

$$\begin{aligned} RHS = &-[\widetilde{W}_{2p-2}, \widetilde{W}_{2K-2p}] - [\widetilde{W}_{2p-2}, W_{2K-2p-2}] \\ &+ [\widehat{B}, W_{2K}] + [\widehat{B}, W_{2K-4}] + \mathrm{d.} \end{aligned} \tag{7.46}$$

The first term can be disposed of in the same fashion as (7.42). The second and fourth are a.s.d. by assumption (7.23). Therefore, recalling the LHS (7.39) one has

$$\begin{aligned} &\tfrac{1}{8}[[B,\widetilde{W}_{2p}],[\widetilde{B},W_{2K-2p-2}]] - \tfrac{1}{8}[[B,\widetilde{W}_{2p-2}],[\widetilde{B},W_{2K-2p}]] \\ &= [\widehat{B},W_{2K}] + [B,\widetilde{W}_{2K}] \quad . \end{aligned} \tag{7.47}$$

If one adds this equation for successive p's: p=1, p=2, etc. up to p=q and uses (7.35) for p=0, m=K, one gets

$$\tfrac{1}{8}[[B,\widetilde{W}_{2q}],[\widetilde{B},W_{2K-2q-2}]] = (q+1)\left([\widetilde{B},W_{2K}]+[B,\widetilde{W}_{2K}]\right) \tag{7.48}$$

which is (7.35) for m=K. Thus the proof by induction of (7.35) is complete.

Returning to the main proof, I am considering the case when N is even. As the additional equation from the set (7.17) with n=N+1, for

the present case I choose to prove

$$[\tilde{B}, W_{2N}] = -[B, \tilde{W}_{2N}] \tag{7.49}$$

i.e. $[\tilde{B}, W_{2N}]$ is a.s.d. This is linearly independent of the equations (7.28). Consider the quantity $[\tilde{W}_n, W_n]$, which is a.s.d. by construction.

$$[\tilde{W}_N, W_N] = -\frac{1}{8}[[\tilde{B},[B,\tilde{W}_{N-2}]], W_N] - [W_{N-2}, W_N] \tag{7.50}$$

The second term can be reduced to $-[\tilde{B},B]$ by assumption (7.23) with n=1, $\ell = \frac{1}{2}N,\ \frac{1}{2}N-1,\ \ldots\ ,\ 1$. One can therefore write

$$-\frac{1}{8}[[\tilde{B},[B,\tilde{W}_{N-2}]], W_N] = \text{a.s.d} \tag{7.51}$$

$$\frac{1}{8}[[B,\tilde{W}_{N-2}],[\tilde{B},W_N]] - \frac{1}{8}[\tilde{B},[[B,\tilde{W}_{N-2}],W_N]] = \text{a.s.d.} \tag{7.52}$$

The second term is

$$-\frac{1}{8}[\tilde{B},[[B,\tilde{W}_{N-2}],W_N]] = -\frac{1}{8}[\tilde{B}[B,[\tilde{W}_{N-2},W_N]]] + \frac{1}{8}[\tilde{B},[\tilde{W}_{N-2},[B,W_N]]]$$

$$= -\frac{1}{8}[\tilde{B},[B,[\tilde{B},W_{2N-2}]]] + \frac{1}{8}[[\tilde{B},\tilde{W}_{N-2}],[B,W_N]] + \frac{1}{8}[\tilde{W}_{N-2},[\tilde{B}[B,W_N]]]$$

$$= [\tilde{B},W_{2N}] + [\tilde{B},\tilde{W}_{2N-2}] + \frac{1}{8}[[\tilde{B},W_{N-4}],[B,\tilde{W}_{N-2}]] - [\tilde{W}_{N-2},\tilde{W}_N] - [\tilde{W}_{N-2},W_{N-2}]. \tag{7.53}$$

With use of (7.35) with 2p=N-4, m=N-2, on the third term of (7.53) and (7.23) again on the last two, (7.53), except for the first term, can be shown to be a.s.d. Thus, returning to (7.52), one finds

$$\frac{1}{8}[[B,\tilde{W}_{N-2}],[\tilde{B},W_N]] + [\tilde{B},W_{2N}] = \text{a.s.d.} \tag{7.54}$$

The first term can be reduced with the help of (7.35) with 2p=N-2, m=N.

$$(\tfrac{1}{2}N)\left([B,\tilde{W}_{2N}] + [\tilde{B},W_{2N}]\right) + [\tilde{B},W_{2N}] = \text{a.s.d.} \tag{7.55}$$

Adding to (7.55) its dual, one obtains

$$(N+1)\left([B,\widetilde{W}_{2N}]+[\widetilde{B},W_{2N}]\right)=0\,. \tag{7.56}$$

Thus (7.49) has been proven. Together with (7.28), (7.49) implies (7.17) with n=N+1, $1\leq \ell \leq N$, N even. Part I is now complete.

Part II. I now extend this result for n=N+1 to include $\ell=0$ and $\ell=N+1$. Since (7.17) with $\ell=N+1$ is just the dual of (7.17) with $\ell=0$, there is only one equation to prove, namely n=N+1, $\ell=0$, i.e.

$$[B,W_{2N+2}] = -[\widetilde{B},W_{2N}] \tag{7.57}$$

or, with the result just established

$$[B,W_{2N+2}] = [B,\widetilde{W}_{2N}]\,. \tag{7.58}$$

I expand $[B, W_{2N+2}]$ in the by now familiar manner.

$$\begin{aligned}
[B,W_{2N+2}] &= -\tfrac{1}{8}[B,[B,[\widetilde{B},W_{2N}]]]-[B,\widetilde{W}_{2N}]\\
&= -\tfrac{1}{8}[B,[[B,\widetilde{B}],W_{2N}]]-\tfrac{1}{8}[B,[\widetilde{B},[B,W_{2N}]]]-[B,\widetilde{W}_{2N}]\\
&= \tfrac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]]+\tfrac{1}{8}[B,[[B,\widetilde{B}],\widetilde{W}_{2N-2}]]\\
&\quad +[B,W_{2N-2}]
\end{aligned} \tag{7.59}$$

where (7.23) has been used for n=M, $\ell=0$, i.e.

$$[B,W_{2N}] = [B,\widetilde{W}_{2N-2}]\,. \tag{7.60}$$

The first term on the right hand side of (7.59) is

$$\frac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]] = \frac{1}{64}[[B,[B,\widetilde{B}]],[B,[\widetilde{B},W_{2N-2}]]]$$

$$+\frac{1}{64}[[B,\widetilde{B}],[B,[B,[\widetilde{B},W_{2N-2}]]]]$$

$$= -\frac{1}{64}[[B,[B,[B,\widetilde{B}]]],[\widetilde{B},W_{2N-2}]] + \frac{1}{64}[B,[[B,[B,\widetilde{B}]],[\widetilde{B},W_{2N-2}]]]$$

$$-\frac{1}{8}[[B,\widetilde{B}],[B,W_{2N}]] - \frac{1}{8}[[B,\widetilde{B}],[B,\widetilde{W}_{2N-2}]]$$

$$= -\frac{1}{4}[[B,\widetilde{B}],[\widetilde{B},W_{2N-2}]] + \frac{1}{64}[B,[B,[[B,\widetilde{B}],[\widetilde{B},W_{2N-2}]]]]$$

$$-\frac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]] - \frac{1}{4}[[B,\widetilde{B}],[B,\widetilde{W}_{2N-2}]]$$

$$= \frac{1}{64}[B,[B,(-[B,[\widetilde{B},[B,\widetilde{W}_{2N-2}]]] - [\widetilde{B},[B,[\widetilde{B},W_{2N-2}]]])]]$$

$$-\frac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]]$$

$$= \frac{1}{8}[B,[B,([B,\widetilde{W}_{2N}]+[B,W_{2N-2}]+[\widetilde{B},W_{2N}]+[\widetilde{B},\widetilde{W}_{2N-2}])]]$$

$$-\frac{1}{64}[B,[[B,\widetilde{B}],[B,[\widetilde{B},W_{2N-2}]]]]. \tag{7.61}$$

The first term in (7.61) is zero due to the assumption (7.23) and (7.56). Thus the expression on the left hand side of (7.61) is equal

to its negative and is therefore zero. Returning to (7.59),

$$[B, W_{2N+2}] = \tfrac{1}{8}[B,[B,[\tilde{B},\tilde{W}_{2N-2}]]] - \tfrac{1}{8}[B,[\tilde{B},[B,\tilde{W}_{2N-2}]]]$$
$$+ [B, W_{2N-2}]$$
$$= -[B,\tilde{W}_{2N-4}] + [B,\tilde{W}_{2N}] + [B, W_{2N-2}]$$
$$= [B,\tilde{W}_{2N}]. \tag{7.62}$$

Thus (7.58) is true and (7.17) holds for n=N+1 and ℓ=0 or ℓ=N+1.

Part III. The final step involved is to raise the limit on ℓ for n=N-1, n=N-3, ... Commute $-\frac{1}{8}[B,\tilde{B}]$ with (7.17) with n=N, ℓ=N to obtain (7.27) with ℓ=N.

$$[\tilde{W}_{2N+2},\tilde{B}] - [\tilde{W}_{2N-2},\tilde{B}] + [\tilde{W}_{2N}, B] + [\tilde{W}_{2N},\tilde{W}_2]$$
$$= -[\tilde{W}_{2N}, B] + [\tilde{W}_{2N-4}, B] + [\tilde{W}_{2N-2}, W_2] + [\tilde{W}_{2N-2},\tilde{B}]. \tag{7.63}$$

Using the results of the previous sections ((7.17) with n=N+1, ℓ=N+1 and ℓ=N) and the assumption (7.23) with n=N-1, ℓ=N-1, one finds that

$$[\tilde{W}_{2N},\tilde{W}_2] = [\tilde{W}_{2N-2},\tilde{B}] \tag{7.64}$$

which is (7.17) with n=N-1, ℓ=N. This is the desired result for n=N-1. The same procedure can be applied to the lower levels, namely, if one commutes $-\frac{1}{8}[B,\tilde{B}]$ with (7.17) with n=k≡N-2j for integral j, 0≤2j≤N-1, and for $\ell=\frac{1}{2}(N+k)$=N-j one gets (7.27) with N→k, $\ell\to\frac{1}{2}(N+k)$:

$$-[\tilde{W}_{N+k+2}, W_{k-N-2}] + [\tilde{W}_{N+k-2}, W_{k-N-2}] + [\tilde{W}_{N+k}, W_{k-N}]$$
$$-[\tilde{W}_{N+k}, W_{k-N-4}] = -[\tilde{W}_{N+k}, W_{k-N}] + [\tilde{W}_{N+k-4}, W_{k-N}]$$
$$+[\tilde{W}_{N+k-2}, W_{k-N+2}] - [\tilde{W}_{N+k-2}, W_{k-N-2}]. \tag{7.65}$$

The right hand side can be seen to be zero from the assumption (7.23) with n=k+1, ℓ=N-j and n=k-1, ℓ=N-j-1, leaving

$$[\tilde{W}_{N+k+2}, W_{k-N-2}] - [\tilde{W}_{N+k}, W_{k-N}] = -[\tilde{W}_{N+k}, W_{k-N-4}] - [\tilde{W}_{N+k-2}, W_{k-N-2}]. \tag{7.66}$$

The left hand side of (7.66) is zero if (7.17) with n=k+1, $\ell = \frac{1}{2}(N+k)+1 = \frac{1}{2}(N+1+(k+1))$ is true. The right hand side is zero if (7.17) with n=k-1, $\ell = \frac{1}{2}(N+k) = \frac{1}{2}(N+1+(k-1))$ holds. If one adds (7.66) times $(-1)^j$ for j=0,1,2,...., q successively (recall k=N-2j) one obtains

$$\begin{aligned} &-[\tilde{W}_{2N+2}, \tilde{B}] - [\tilde{W}_{2N}, B] \\ &= (-1)^q\left(-[\tilde{W}_{2N-2q}, W_{-2q-4}] + [\tilde{W}_{2N-2q-2}, W_{-2q-2}]\right) \end{aligned} \tag{7.67}$$

for $0 \le q \le \frac{1}{2}(N-1)$. The left side is zero by (7.62). Thus

$$[\tilde{W}_{2N-2q}, W_{-2q-4}] = [\tilde{W}_{2N-2q-2}, W_{-2q-2}] \tag{7.68}$$

for $0 \le q \le \frac{1}{2}(N-1)$. Setting n=N-2q-1, ℓ=N-q in (7.68) yields (7.17) for n=N-2q-1, $\ell = \frac{1}{2}(N+n+1)$:

$$[\tilde{W}_{2\ell}, W_{2n-2\ell-2}] = [\tilde{W}_{2\ell-2}, W_{2n-2\ell}]. \tag{7.69}$$

Looking at (7.24) one can see that the desired result has been achieved, i.e. the raising of the integral limit on ℓ by one for n=N-1, N-3, ..., (0 or 1). This is equivalent to raising the half-integral limit on ℓ given by $\ell = \frac{1}{2}(N+n)$ by $\frac{1}{2}$ for all n≤N. All entries in (7.24) have now been derived. The proof by induction of (7.17) is complete.

Thus, (7.17) holds for all n and ℓ and charges Q_{2n} defined in (7.2) are conserved for all n (n=1,2,3,...). This result requires only two assumptions. The first is that the Hamiltonian can be

written in the self-dual form (7.1). The second assumption is (7.8), which states that the first charge in the set, i.e. Q_2, is conserved.

In addition to the conservation of charges $[Q_{2n},H]=0$, it is also possible to show that the charges Q_{2n} commute with each other. $[Q_{2n},Q_{2m}]$ will vanish if

$$[(W_{2n}-\tilde{W}_{2n-2}),(W_{2m}-\tilde{W}_{2m-2})]=0 \tag{7.70}$$

and

$$[(W_{2n}-\tilde{W}_{2n-2}),(\tilde{W}_{2m}-W_{2m-2})]=a.s.d. \tag{7.71}$$

Consider $[W_{2n},W_{2m}]$. By (7.16) and (7.17)

$$[W_{2n},W_{2m}]=-[W_{2n},\tilde{W}_{-2m-2}]=-[B,\tilde{W}_{2n-2m-2}]. \tag{7.72}$$

Also,

$$[\tilde{W}_{2n-2},\tilde{W}_{2m-2}]=-[\tilde{W}_{2n-2},W_{-2m}]=-[B,W_{2n-2m-2}]$$
$$=[B,\tilde{W}_{2n-2m-2}]. \tag{7.73}$$

So,

$$[W_{2n},W_{2m}]=-[\tilde{W}_{2n-2},\tilde{W}_{2m-2}]. \tag{7.74}$$

By a similar argument (7.17) shows that

$$[W_{2n},\tilde{W}_{2m-2}]=[W_{2n-2},\tilde{W}_{2m}]=-[\tilde{W}_{2n-2},W_{2m}]. \tag{7.75}$$

Together (7.74) and (7.75) prove (7.70). It can easily be checked that (7.17) implies that all four of the commutators in (7.71) are a.s.d., thus (7.71) is satisfied. Therefore

$$[Q_{2n}, Q_{2m}] = 0 \tag{7.76}$$

for all n,m, $1 \leq n \leq \infty$, $1 \leq m \leq \infty$.

VIII Applications

As I have mentioned, the result of the previous chapter has potential applicability to a wide variety of theories, since it is a general operator statement and does not refer to the number of space-time dimensions or the precise nature of the space-time manifold (i.e. lattice, continuum, or loop space).[34] There are many interesting theories which are self-dual. Unfortunately, however, the additional condition of the theorem

$$[B,[B,[B,\tilde{B}]]] \propto [B,\tilde{B}] \tag{8.1}$$

appears to be stronger than one might have hoped. The condition is satisfied in the case of the XZ model if one takes

$$B = \sum_J \sigma^1_J \sigma^1_{J+1} \quad , \quad \tilde{B} = \sum_J \sigma^3_J \sigma^3_{J+1} \quad . \tag{8.2}$$

The charges generated by the theorem (7.2) are identical to the even-numbered charges found by the more heuristic procedure of Chap. V (5.19). As discussed in Chap. VI these are the same charges that result from the complete quantum integrability of the system. Note, however, that only half of the complete set is given by the theorem. The odd-numbered charges - those which were independent of the coupling constants - are not produced. This is because the construction procedure is specifically designed to produce charges with a non-trivial coupling constant dependence.

Two other models have been found which satisfy the condition (8.1). The first is the one-dimensional quantum Ising model in an external magnetic field. The second is the generalized XZ model as defined by Suzuki.[27] The Ising model Hamiltonian

$$H_I = \sum_J (\Gamma\, \sigma^3_J \sigma^3_{J+1} + k\, \sigma^1_J) \tag{8.3}$$

is self-dual if one takes

$$B = \sum_J \sigma_J^1 \quad , \quad \tilde{B} = \sum_J \sigma_J^3 \sigma_{J+1}^3 \tag{8.4}$$

under the dual transformation given by (4.12) (see also App. B). Then

$$[B,\tilde{B}] = (2i) \sum_J \left(-\sigma_J^2 \sigma_{J+1}^3 - \sigma_J^3 \sigma_{J+1}^2\right)$$

$$[B,[B,\tilde{B}]] = (2i)^2 \sum_J \left(-2\sigma_J^3 \sigma_{J+1}^3 + 2\sigma_J^2 \sigma_{J+1}^2\right)$$

$$[B,[B,[B,\tilde{B}]]] = (2i)^3 \sum_J \left(4\sigma_J^2 \sigma_{J+1}^3 + 4\sigma_J^3 \sigma_{J+1}^2\right) = 16[B,\tilde{B}], \tag{8.5}$$

so

$$\begin{aligned} W_2 &= -\tfrac{1}{8}[B,[\tilde{B},B]] - \tilde{B} = \sum_J \left(\sigma_J^3 \sigma_{J+1}^3 - \sigma_J^2 \sigma_{J+1}^2\right) - \sum_J \sigma_J^3 \sigma_{J+1}^3 \\ &= -\sum_J \sigma_J^2 \sigma_{J+1}^2 \end{aligned} \tag{8.6}$$

$$\begin{aligned} \tilde{W}_2 &= -\tfrac{1}{8}[\tilde{B},[B,\tilde{B}]] - B = \sum_J \left[\left(\sigma_J^3 \sigma_{J+1}^1 \sigma_{J+2}^3 + \sigma_J^1\right) - \sigma_J^1\right] \\ &= \sum_J \sigma_J^3 \sigma_{J+1}^1 \sigma_{J+2}^3 \end{aligned} \tag{8.7}$$

$$Q_2 = k\sum_J \left(-\sigma_J^2 \sigma_{J+1}^2 - \sigma_J^3 \sigma_{J+1}^3\right) + \Gamma \sum_J \left(\sigma_J^3 \sigma_{J+1}^1 \sigma_{J+2}^3 - \sigma_J^1\right). \tag{8.8}$$

The conservation of this charge can be checked explicitly (the theorem, of course, guarantees its conservation). If one calculates several higher charges, which is very easy since all that is involved is the the calculation of two commutators, a pattern quickly develops which allows one to induce the form of the N^{th} charge. It is given by

$$Q_{2N} = k\sum_J \left(-\sigma_J^2\sigma_{J+1}^1\sigma_{J+2}^1\cdots\sigma_{J+N-1}^1\sigma_{J+N}^2 - \sigma_J^3\sigma_{J+1}^1\cdots\sigma_{J+N-1}^1\sigma_{J+N}^3\right)$$
$$+\Gamma\sum_J\left(\sigma_J^3\sigma_{J+1}^1\sigma_{J+2}^1\cdots\sigma_{J+N}^1\sigma_{J+N+1}^3 + \sigma_J^2\sigma_{J+1}^1\cdots\sigma_{J+N-2}^1\sigma_{J+N-1}^2\right) \tag{8.9}$$

These are similar in structure to the XZ model charges (5.19) but there are some significant differences, e.g. the Ising model charges involve strings of three different lengths whereas the XZ model charges are made up of only two different string lengths. A much deeper connection exists, however, than might be apparent at first glance. This is because there exists a transformation which relates the XZ model to the sum of two independent Ising models. This transformation is given in App. B and allows one to derive the Ising model charges from the XZ model charges and vice versa.

The Hamiltonian of the generalized XZ model is given by:

$$H_r = \sum_J\left(J_x\sigma_J^1\sigma_{J+1}^2\sigma_{J+2}^2\cdots\sigma_{J+r}^2\sigma_{J+r+1}^1 + J_z\sigma_J^3\sigma_{J+1}^2\sigma_{J+2}^2\cdots\sigma_{J+r}^2\sigma_{J+r+1}^3\right) \tag{8.10}$$
$$r = 0, 1, 2, \cdots$$

It can easily be checked that it saitsfies (8.1) and a previously unknown set of conserved charges can be generated for this model. The full XYZ model can also be written in a self-dual form:

$$H_{xyz} = kB + \Gamma\widetilde{B} \tag{8.11}$$

where

$$B = \sum_J \sigma_J^1\sigma_{J+1}^1 + \lambda\sum_J\sigma_J^2\sigma_{J+1}^2 \tag{8.12}$$

$$\widetilde{B} = \sum_J \sigma_J^3\sigma_{J+1}^3 + \lambda\sum_J\sigma_J^2\sigma_{J+1}^2 \tag{8.13}$$

$$k = -\tfrac{1}{2} J_x , \quad \Gamma = -\tfrac{1}{2} J_z , \quad \lambda = J_y / (J_x + J_z) \tag{8.14}$$

and the dual transformation is the XZ dual transformation $\sigma^1 \leftrightarrow \sigma^3$ (If $J_X = -J_Z$ then one can take instead the XY or YZ dual transformations). Under this decomposition, however, the XYZ model does not meet the condition (8.1). The problem is that higher powers of λ enter the higher commutators preventing the possibility of repetition. It may be that some other decomposition exists which does satisfy (8.1). It might also be that the symmetry one needs to exploit in the case of the XYZ model is triality,

$$\begin{tikzcd}[row sep=small] & \sigma_J^3 \arrow[dl] \\ \sigma_J^1 \arrow[dr] & \\ & \sigma_J^2 \arrow[uu] \end{tikzcd} \tag{8.15}$$

rather than duality. In this case one would need to formulate an analogous theorem for self-trial Hamiltonians:

$$H = kB + \Gamma \tilde{B} + \delta \tilde{\tilde{B}} \tag{8.16}$$

where $\sim$ now indicates the triality transformation. Such a treatment would put all of the coupling constants on an equal footing. However, this approach may be difficult since, as is apparent from the calculation of C_2 for the XYZ model (App. A), the coefficients of terms in the higher conserved charges are very complicated polynominals in the coupling constants.

I also checked the condition (8.1) on two other self-dual models: the Z(N) spin theory in two dimensions, and the Z(2) gauge theory in four dimensions. The Z(N) spin theory is a generalization of the Ising model where the variable can take on one of N states as opposed to just two. It is the quantum model associated with the classical two-dimensional Potts model.[25] The Z(2) Gauge Theory in four di-

mensions[12] is an interesting one to study since Z(2) is the center of the group SU(2). Its Hamiltonian is given by

$$H = B + \lambda \tilde{B} , \tag{8.17}$$

where

$$B = \sum_{\vec{n},\hat{\ell}} \sigma^1(\vec{n},\hat{\ell}) , \tag{8.18}$$

$$\tilde{B} = \tfrac{1}{2} \sum_{\vec{n},\hat{\ell},\hat{k}\neq\hat{\ell}} \sigma^3(\vec{n},\hat{\ell})\, \sigma^3(\vec{n}+\hat{\ell},\hat{k})\, \sigma^3(\vec{n}+\hat{k},\hat{\ell})\, \sigma^3(\vec{n},\hat{k}) , \tag{8.19}$$

and $\vec{n}$ is a displacement vector which labels sites on the four-dimensional lattice. $\hat{\ell}$ and $\hat{k}$ are unit lattice vectors which are summed over the four positive lattice axis directions. The spins for this model reside on the links of the lattice. The local gauge transformation under which the Hamiltonian is invariant consists of flipping all of the eight spins attached to a single lattice site $\vec{n}$ which is accomplished by applying the operator

$$G(\vec{n}) = \prod_{\hat{\ell},\, \epsilon=\pm 1} \sigma^1(\vec{n}, \epsilon\hat{\ell}) . \tag{8.20}$$

A positive result for Z(2) would give one hope that the procedure might also be successful for the SU(2) Yang-Mills theory. Unfortunately the condition (8.1) is not satisfied by either the Z(N) spin or Z(2) gauge theories considered. Since it is not known how to formulate the SU(2) theory in a fully self-dual manner the theorem cannot be applied to it at this time. An attempt was made to test the condition for the A and B operators of 't Hooft,[13] where

$$A(C) = \tfrac{1}{2}\, \mathrm{tr}\, P e^{i\oint_C \vec{A}\cdot d\vec{x}} , \tag{8.21}$$

C is some path in space-time, and B(C') is defined so that

$$A(C)\,B(C') = B(C')\,A(C)\,e^{\frac{2\pi i n}{N}} . \tag{8.22}$$

N is from SU(N) and n is the number of times the path C encircles C'. It is hoped that a non-local reformulation of the Yang-Mills theory can be achieved through the use of these operators. B(C) can also be obtained from A(C) through a partial Kramers-Wannier dual transformation, by expressing elements of SU(N) in terms of products of elements of SU(N)/Z(N) and Z(N). A transformation similar to Kramers-Wannier duality can then be performed on the Z(N) part only which results in B (C).[14] One finds that

$$[A(C),[A(C),[A(C),B(C')]]] = A^2(C)\left(1-e^{\frac{2\pi i n}{N}}\right)^2[A(C),B(C')] , \tag{8.23}$$

so the condition is not satisfied. Even if it were satisfied, however, it is not completely clear what it would mean, since one does not yet know how to express the Hamiltonian in terms of A(C) and B(C). It would yield a set of commuting, but not necessarily conserved operators. Note also that here we are dealing with operators defined on specific paths whereas previously the operators were summed over position and therefore translationally invariant. To date I have found no other models which satisfy the condition of the theorem, although the possibilities have by no means been completely exhausted.

IX Conclusion

The objective of this thesis was to investigate the possible connection between the property of self-duality in a quantum field theory and the presence of an infinite set of conserved charges. Such a connection was motivated by the similarity of approaches to the search for both a dual transformation and infinite sets of conserved charges for the four-dimensional Yang-Mills theory. Success in either of these ventures would provide valuable non-perturbative information toward a solution of that theory. A definite connection between these ideas would be of help in advancing this program.

I first attempted to establish such a connection in simpler two-dimensional spin theories. To this end I found the explicit form of an infinite set of charges for two simple self-dual quantum models - the XY and Ising models. These examples led to the formulation of an infinite set of charges for the general self-dual quantum Hamiltonian (linear in the coupling constant). It was proven that these charges form a conserved commuting set in the general case , provided only that the first charge in the set was conserved. The assumption of self-duality played an important role in this proof. Essentially it provides all of the necessary information to guarantee the conservation of the higher charges given that the first charge is conserved. Thus for a class of models (i.e. those whose first charges are conserved) a connection between self-duality and infinite sets of conserved charges was established. This is the first time such a definite connection between these concepts has been found. It suggests that Kramers-Wannier self-duality may be an important property of exactly integrable quantum systems. The relevant class of models is not limited to the two-dimensional lattice models. The theorem is applicable to both higher dimensional and continuum theories. Unfortunately, no examples have been found which satisfy the necessary condition of the theorem outside the realm of the two-dimensional spin theories. It is not clear whether the original goal of establishing a relationship between self-duality and infinite sets of conserved charges for the four-dimensional

gauge theories will be met by the connection found here. This connection was based upon a set of charges generated by a particular commutator algebra. Undoubtedly there are other sets which are generated by other algebras. A much more sophisticated algebra may be needed to generate conserved charges for the four-dimensional theory. Thus one may hope that the connection established here, if not itself applicable to the physically relevant gauge systems, will nevertheless point the way to a method by which a similar connection might be found for those theories.

Appendix A

Calculation of C_2 for the XYZ Model

In this section I give the detailed calculation of the charge C_2 for the XYZ model from the generating function (3.22) derived from the 8-vertex model transfer matrix. The main purpose of this exercise is to show that C_2 as given by (3.22) is the same charge as is given by the heuristic procedure of Chap. V (5.10) or the theorem of Chap. VII (7.2) for the simpler case of the XZ model. It happens that the simplifying assumption $J_y \propto p_2^{\prime}=0$ does not simplify the calculation of C_2 from (3.22) very much. Thus it is reasonable to compute the XYZ charge first and then set $J_y=0$. The exact form of C_2 may be of use in finding an alternate self-dual decomposition for H_{XYZ} that would satisfy the condition of the theorem (7.8).

By differentiating the logarithm in (3.22) and using (3.20) one finds that

$$C_2 = G_2 - 3G_1G_0 + 2G_0^3 \tag{A.1}$$

where

$$G_0 = \frac{1}{2}\sum_J P_i^{\prime}\sigma_J^i\sigma_{J+1}^i \tag{A.2}$$

$$G_1 = \frac{1}{2}\sum_{J<k} P_i^{\prime}P_j^{\prime}\sigma_J^i\sigma_{J+1}^i\sigma_k^j\sigma_{k+1}^j + \frac{1}{2}\sum_J P_i^{\prime\prime}\sigma_J^i\sigma_{J+1}^i \tag{A.3}$$

$$\begin{aligned} G_2 = {} & \frac{3!}{8}\sum_{J<k<l} P_i^{\prime}P_j^{\prime}P_m^{\prime}\sigma_J^i\sigma_{J+1}^i\sigma_k^j\sigma_{k+1}^j\sigma_l^m\sigma_{l+1}^m && (1) \\ & + \frac{3}{4}\sum_{J<k} P_i^{\prime\prime}P_j^{\prime}\sigma_J^i\sigma_{J+1}^i\sigma_k^j\sigma_{k+1}^j && (2) \\ & + \frac{3}{4}\sum_{J<k} P_i^{\prime}P_j^{\prime\prime}\sigma_J^i\sigma_{J+1}^i\sigma_k^j\sigma_{k+1}^j && (2) \\ & + \frac{1}{2}\sum_J P_i^{\prime\prime\prime}\sigma_J^i\sigma_{J+1}^i && (4) \end{aligned} \tag{A.4}$$

and there is an implied sum from one to four of all lower case repeated indices (i,j,m).

$$G_1 G_0 = \frac{1}{4}\Big(\sum_{J<K<L} + \sum_{J<L<K} + \sum_{L<J<K}\Big) P_i' P_j' P_m' \sigma^i_J \sigma^i_{J+1} \sigma^j_K \sigma^j_{K+1} \sigma^m_L \sigma^m_{L+1} \qquad (1)$$

$$+ \frac{1}{4} \sum_{J<K} P_i' P_j' P_m' \sigma^i_J \sigma^i_{J+1} \sigma^j_K \sigma^j_{K+1} \sigma^m_K \sigma^m_{K+1} \qquad (3)$$

$$+ \frac{1}{4} \sum_{J<K} P_i' P_j' P_m' \sigma^i_J \sigma^i_{J+1} \sigma^j_K \sigma^j_{K+1} \sigma^m_J \sigma^m_{J+1} \qquad (3)$$

$$+ \frac{1}{4} \sum_{J<K} P_i'' P_j' \sigma^i_J \sigma^i_{J+1} \sigma^j_K \sigma^j_{K+1} \qquad (2)$$

$$+ \frac{1}{4} \sum_{J<K} P_i' P_j'' \sigma^j_K \sigma^j_{K+1} \sigma^i_J \sigma^i_{J+1} \qquad (2)$$

$$+ \frac{1}{4} \sum_{J} P_i'' P_j' \sigma^i_J \sigma^i_{J+1} \sigma^j_J \sigma^j_{J+1} \qquad (4)$$

(A.5)

$$G_0^3 = \frac{1}{8} \sum_{J,K,L} P_i' P_j' P_m' \sigma^i_J \sigma^i_{J+1} \sigma^j_K \sigma^j_{K+1} \sigma^m_L \sigma^m_{L+1}$$

$$\sum_{J,K,L} = \underbrace{\sum_{\substack{J<K<L \\ K<J<L \\ L<K<J \\ J<L<K \\ K<L<J \\ L<J<K}}}_{(1)} + \underbrace{\sum_{\substack{J=K<L \\ J=L<K \\ K=L<J \\ J=K>L \\ J=L>K \\ K=L>J}}}_{(3)} + \underbrace{\sum_{J=K=L}}_{(4)}$$

(A.6)

Note I have labelled sets of terms by (1), (2), (3), (4) in order to separate like quantities for further treatment. Consider all of the terms of C_2 labelled (1), those which involve four sites or more. First, put all sums in canonical order $\sum_{J<K<L}$ by labelling J,K, and L appropriately in each sum. Then

$$\begin{aligned}
(1) &= \frac{3}{4} JKL - \frac{3}{4}(JKL + JLK + KLJ) \\
&\quad + \frac{1}{4}(JKL + KJL + LJK + JLK + KLJ + LKJ) \\
&= -\frac{1}{2}(JLK + KLJ) + \frac{1}{4}(JKL + KLJ + JLK + LKJ) \\
&= \frac{1}{4}\left([J,[K,L]]\right) = \frac{1}{4}\sum_{J<K<L} P_i' P_j' P_m' \left[\sigma_J^i \sigma_{J+1}^i, \left[\sigma_K^j \sigma_{K+1}^j, \sigma_L^m \sigma_{L+1}^m\right]\right]
\end{aligned} \tag{A.7}$$

where the notation KLJ means

$$KLJ \equiv \sum_{J<K<L} P_i' P_j' P_m' \, \sigma_K^i \sigma_{K+1}^i \, \sigma_L^j \sigma_{L+1}^j \, \sigma_J^m \sigma_{J+1}^m \tag{A.8}$$

The terms labelled (2) are given by

$$\begin{aligned}
(2) &= \frac{3}{4}\sum_{J<K} P_i'' P_j' \, \sigma_J^i \sigma_{J+1}^i \, \sigma_K^j \sigma_{K+1}^j \\
&\quad + \frac{3}{4}\sum_{J<K} P_i' P_j'' \, \sigma_J^i \sigma_{J+1}^i \, \sigma_K^j \sigma_{K+1}^j \\
&\quad - \frac{3}{4}\sum_{J<K} P_i'' P_j' \, \sigma_J^i \sigma_{J+1}^i \, \sigma_K^j \sigma_{K+1}^j \\
&\quad - \frac{3}{4}\sum_{J<K} P_i' P_j'' \, \sigma_K^j \sigma_{K+1}^j \, \sigma_J^i \sigma_{J+1}^i \\
&= \frac{3}{4}\sum_{J<K} P_i' P_j'' \left[\sigma_J^i \sigma_{J+1}^i, \sigma_K^j \sigma_{K+1}^j\right]
\end{aligned} \tag{A.9}$$

For the next group ③ , we must again put the sums in a canonical order $\sum_{J<K}$.

$$③ = -\frac{3}{4}\sum_{J<k}(Jkk + JkJ)$$
$$+\frac{1}{4}\sum_{J<k}(JJk + JkJ + kJJ + kkJ + kJk + Jkk)$$
$$= \frac{1}{4}\sum_{J<k}\left([k,J]k + [kk,J] + [J,[J,k]]\right)$$

(A.10)

$$③ = \frac{1}{4}\sum_{J<K} P_i'P_j'P_m'\Big([\sigma_k^j\sigma_{k+1}^j, \sigma_J^i\sigma_{J+1}^i]\sigma_k^m\sigma_{k+1}^m$$
$$+[\sigma_k^j\sigma_{k+1}^j\sigma_k^m\sigma_{k+1}^m, \sigma_J^i\sigma_{J+1}^i]$$
$$+[\sigma_J^i\sigma_{J+1}^i, [\sigma_J^j\sigma_{J+1}^j, \sigma_k^m\sigma_{k+1}^m]]\Big)$$

(A.11)

Now the commutators are zero if K≠J+1 or i=4. For this term (i.e. ③) $\sum_{ijk}$ will go from one to three only and the cases j=4, k=4 will be treated separately.

$$③ = \frac{i}{2}\sum_{J<k}\left(P_i'P_j'P_m'\epsilon_{jil}\sigma_J^i\sigma_{J+1}^l\sigma_{J+2}^j\sigma_{J+1}^m\sigma_{J+2}^m + P_i'P_j'P_4'\epsilon_{jil}\sigma_J^i\sigma_{J+1}^l\sigma_{J+2}^j\right)$$
$$+\frac{i}{2}\sum_{J<k}P_i'P_j'P_m'\Big(\delta_{im}\epsilon_{jil}\sigma_J^i\sigma_{J+1}^l\sigma_{J+2}^j\sigma_{J+1}^i\sigma_{J+2}^i$$
$$+\delta_{ij}\epsilon_{mil}\sigma_J^i\sigma_{J+1}^i\sigma_{J+2}^i\sigma_{J+1}^l\sigma_{J+2}^m\Big)$$
$$+\frac{i}{2}\sum_{J<k}P_i'P_j'P_4'\, 2\,\epsilon_{jil}\sigma_J^i\sigma_{J+1}^l\sigma_{J+2}^j$$
$$-\sum_{J<k}P_i'P_j'P_m'\epsilon_{jml}\left(\delta_{ij}\epsilon_{ilp}\sigma_{J+1}^p\sigma_{J+2}^m + \delta_{il}\epsilon_{ijp}\sigma_J^p\sigma_{J+2}^m\right)$$

(A.12)

$$③ = \frac{i}{2} \sum_{J<K} P_i' P_j' P_m' \epsilon_{jil} \sigma_J^i \left(\delta_{lm} + i \epsilon_{lma} \sigma_{J+1}^a\right)\left(\delta_{jm} + i \epsilon_{jmb} \sigma_{J+2}^b\right)$$

$$+ \frac{i}{2} \sum_{J<K} P_i' P_j' P_4' \epsilon_{jil} \sigma_J^i \sigma_{J+1}^l \sigma_{J+2}^j$$

$$- i \sum_{J<K} P_i'^2 P_j' \epsilon_{jil} \epsilon_{lia} \epsilon_{jib} \sigma_J^i \sigma_{J+1}^a \sigma_{J+2}^b$$

$$+ i \sum_{J<K} P_i' P_j' P_4' \epsilon_{jil} \sigma_J^i \sigma_{J+1}^l \sigma_{J+2}^j$$

$$+ \sum_{J<K} P_i'^2 P_m' (\epsilon_{iml})^2 \sigma_{J+1}^m \sigma_{J+2}^m$$

$$- \sum_{J<K} P_i' P_j' P_m' (\epsilon_{jmi})^2 \sigma_J^m \sigma_{J+2}^m$$

(A.13)

$$\epsilon_{jil} \epsilon_{lia} \epsilon_{jib} = -\delta_{aj} \epsilon_{jib}$$

(A.14)

$$3^{rd} \text{ term} \rightarrow i \sum_{J<K} P_i'^2 P_j' \epsilon_{jib} \sigma_J^i \sigma_{J+1}^j \sigma_{J+2}^b$$

(A.15)

$$1^{st} \text{ term} = -\frac{1}{2} \sum_{J<K} P_i' P_j' P_m' \epsilon_{jim} \epsilon_{jmb} \sigma_J^i \sigma_{J+2}^b$$

$$- \frac{1}{2} \sum_{J<K} P_i' P_j'^2 \epsilon_{jil} \epsilon_{lja} \sigma_J^i \sigma_{J+1}^a$$

$$- \frac{i}{2} \sum_{J<K} P_i' P_j' P_m' \epsilon_{jil} \epsilon_{lma} \epsilon_{jmb} \sigma_J^i \sigma_{J+1}^a \sigma_{J+2}^b$$

(A.16)

$$\epsilon_{jil} \epsilon_{lma} \epsilon_{jmb} = \left(\delta_{jm} \delta_{ia} - \delta_{ja} \delta_{im}\right) \epsilon_{jmb} = -\delta_{ja} \delta_{im} \epsilon_{jib}$$

(A.17)

$$
\begin{aligned}
③ = & \tfrac{1}{2} \sum_J P_i' P_j' P_m' (\epsilon_{jim})^2 \sigma_J^i \sigma_{J+2}^i \\
& - \tfrac{1}{2} \sum_J P_i' P_j'^2 (\epsilon_{jil})^2 \sigma_J^i \sigma_{J+1}^i \\
& + \tfrac{i}{2} \sum_J P_i'^2 P_j' \epsilon_{jib} \sigma_J^i \sigma_{J+1}^j \sigma_{J+2}^b \\
& + \tfrac{i}{2} \sum_J P_i' P_j' P_4' \epsilon_{jil} \sigma_J^i \sigma_{J+1}^l \sigma_{J+2}^j \\
& + i \sum_J P_i'^2 P_j' \epsilon_{jib} \sigma_J^i \sigma_{J+1}^j \sigma_{J+2}^b \\
& + i \sum_J P_i' P_j' P_4' \epsilon_{jil} \sigma_J^i \sigma_{J+1}^l \sigma_{J+2}^j \\
& + \sum_J P_i'^2 P_m' (\epsilon_{iml})^2 \sigma_J^m \sigma_{J+1}^m \\
& - \sum_J P_i' P_j' P_m' (\epsilon_{jmi})^2 \sigma_J^m \sigma_{J+2}^m
\end{aligned}
\tag{A.18}
$$

$$
\begin{aligned}
③ = & -\tfrac{1}{2} \sum_J P_i' P_j' P_m' (\epsilon_{jim})^2 \sigma_J^m \sigma_{J+2}^m \\
& + \tfrac{1}{2} \sum_J P_i' P_j'^2 (\epsilon_{jil})^2 \sigma_J^i \sigma_{J+1}^i \\
& + \tfrac{3i}{2} \sum_J P_i'^2 P_j' \epsilon_{jib} \sigma_J^i \sigma_{J+1}^j \sigma_{J+2}^b \\
& + \tfrac{3i}{2} \sum_J P_i' P_j' P_4' \epsilon_{jil} \sigma_J^i \sigma_{J+1}^l \sigma_{J+2}^j
\end{aligned}
\tag{A.19}
$$

$$③ = -\frac{1}{2}\sum_J P_i' P_j' P_m' \, (\epsilon_{jim})^2 \, \sigma_J^m \sigma_{J+2}^m$$
$$+\frac{1}{2}\sum_J P_i' P_j'^2 \, (\epsilon_{jie})^2 \, \sigma_J^i \sigma_{J+1}^i$$
$$+\frac{3i}{2}\sum_J \left(P_i'^2 P_j' - P_i' P_b' P_4'\right) \epsilon_{jib} \, \sigma_J^i \sigma_{J+1}^j \sigma_{J+2}^b$$

(A.20)

Recall,

$$② = \frac{3i}{2}\sum_J P_i' P_j'' \, \epsilon_{ije} \, \sigma_J^i \sigma_{J+1}^e \sigma_{J+2}^j$$

(A.21)

$$② + ③ = -\frac{1}{2}\sum_J P_i' P_j' P_m' \, (\epsilon_{jim})^2 \sigma_J^m \sigma_{J+2}^m$$
$$+\frac{1}{2}\sum_J P_i' P_j'^2 \, (\epsilon_{jie})^2 \sigma_J^i \sigma_{J+1}^i$$
$$+\frac{3i}{2}\sum_J \left(P_i'^2 P_j' + P_i' P_b'' - P_i' P_b' P_4'\right) \epsilon_{jib} \, \sigma_J^i \sigma_{J+1}^j \sigma_{J+2}^b$$

(A.22)

$$\left(P_i'^2 P_j' + P_i' P_b'' - P_i' P_b' P_4'\right) \epsilon_{jib} = P_i' \left(P_i' P_j' + P_b'' - P_b' P_4'\right) \epsilon_{jib}$$

(NO SUM ON j, i, b)

(A.23)

Evaluate for $i \neq j \neq b \neq i$ e.g. $i=1$, $j=2$, $b=3$.

$$\left(P_1'P_2' + P_3'' - P_3'P_4'\right) = \frac{dn(2J)\,cn(2J)}{sn^2(2J)} - \ell^2 sn^2(J) - \frac{cn(2J)+dn(2J)-1}{sn^2(2J)}$$

$$= \frac{dn(J)cn(J) + dn^2(J)cn(J) - \ell^2 sn^2(J) + \ell^2 cn(J)sn^2(J) - cn(J) + 1 - cn(J)dn(J) - dn^2(J)}{sn^2(J)\,(1+dn(J))}$$

$$= 0 \tag{A.24}$$

So

$$2 + 3 = -\frac{1}{2}\sum_J P_i'P_j'P_m' \,(\epsilon_{jim})^2\, \sigma_J^m \sigma_{J+2}^m$$

$$+ \frac{1}{2}\sum_J P_i'P_j'^2 \,(\epsilon_{jil})^2\, \sigma_J^i \sigma_{J+1}^i \tag{A.25}$$

The final group of terms is 4.

$$4 = \frac{1}{2}\sum_J P_i''' \sigma_J^i \sigma_{J+1}^i - \frac{3}{4}\sum_J P_i''P_j' \sigma_J^i \sigma_{J+1}^i \sigma_J^j \sigma_{J+1}^j$$

$$+ \frac{1}{4}\sum_J P_i'P_j'P_m' \,\sigma_J^i \sigma_{J+1}^i \sigma_J^j \sigma_{J+1}^j \sigma_J^m \sigma_{J+1}^m \tag{A.26}$$

$$4 = \frac{1}{2}\sum_J P_i''' \sigma_J^i \sigma_{J+1}^i$$

$$- \frac{3}{4}\sum_J \left(P_i''P_j' - P_i'P_j'P_4'\right)\left(\delta_{ij} + i\epsilon_{ijl}\sigma_J^l\right)\left(\delta_{ij} + i\epsilon_{ijm}\sigma_{J+1}^m\right)$$

$$- \frac{3}{4}\sum_J \left(P_i''P_4' + P_4''P_i'\right) \sigma_J^i \sigma_{J+1}^i + \frac{3}{4}\sum_J P_i'P_4'^2 \sigma_J^i \sigma_{J+1}^i$$

$$+ \frac{1}{4}\sum_J P_i'P_j'P_m' \left(\delta_{ij} + i\epsilon_{ijk}\sigma_J^k\right)\sigma_J^m \left(\delta_{ij} + i\epsilon_{ijl}\sigma_{J+1}^l\right)\sigma_{J+1}^m \tag{A.27}$$

$$+ \text{const.}$$

$$(\delta_{ij} + i\epsilon_{ijk}\sigma_J^k)\sigma_J^m(\delta_{ij} + i\epsilon_{ijl}\sigma_{J+1}^l)\sigma_{J+1}^m$$

$$= (\delta_{ij}\sigma_J^m + i\epsilon_{ijm} - \epsilon_{ijk}\epsilon_{kma}\sigma_J^a)(\delta_{ij}\sigma_{J+1}^m + i\epsilon_{ijm} - \epsilon_{ijl}\epsilon_{lmb}\sigma_{J+1}^b)$$

(A.28)

$$
\begin{aligned}
④ = {} & \frac{1}{2}\sum_J P_i''' \sigma_J^i \sigma_{J+1}^i \\
& + \frac{3}{4}\sum_J (P_i'' P_j' - P_i' P_j' P_4')(\epsilon_{ijm})^2 \sigma_J^m \sigma_{J+1}^m \\
& - \frac{3}{4}\sum_J (P_i'' P_4' + P_4'' P_i')\sigma_J^i \sigma_{J+1}^i \\
& + \frac{1}{4}\sum_J P_i'^2 P_m' \sigma_J^m \sigma_{J+1}^m \\
& + \frac{1}{2}\sum_J (P_i'^2 P_j' \sigma_J^j \sigma_{J+1}^j - P_i'^3 \sigma_J^i \sigma_{J+1}^i) \\
& + \frac{3}{4}\sum_J P_i' P_4'^2 \sigma_J^i \sigma_{J+1}^i
\end{aligned}
$$

(A.29)

Finally,

$$
\begin{aligned}
① + ② + ③ + ④ = C_2 = &-\sum_J P_i' P_j' P_m' \epsilon_{ija} \epsilon_{jmb} \sigma_J^i \sigma_{J+1}^a \sigma_{J+2}^b \sigma_{J+3}^m \\
&- \frac{1}{2} \sum_J P_i' P_j' P_m' (\epsilon_{jim})^2 \sigma_J^i \sigma_{J+2}^i \\
&+ \frac{1}{2} \sum_J \Big(P_i''' + \frac{3}{2} (\epsilon_{jim})^2 P_j'' P_m' - \frac{3}{2} \delta_{im} \delta_{ij} P_i'' P_4' - \frac{3}{2} \delta_{im} \delta_{ij} P_i' P_4'' \\
&+ \frac{3}{2} \delta_{im} P_j'^2 P_i' - \frac{3}{2} (\epsilon_{jmi})^2 P_j' P_m' P_4' - \delta_{im} \delta_{ij} P_i'^3 + (\epsilon_{jim})^2 P_i' P_j'^2 \\
&+ \frac{3}{2} \delta_{ij} \delta_{im} P_i' P_4'^2 \Big) \sigma_J^i \sigma_{J+1}^i
\end{aligned}
$$

(implied sum on i, j, m from 1 to 3). (A.30)

Of course the calculation is not really finished at this point. If one is given a Hamiltonian with coupling constants (J_X, J_Y, J_Z) and asked to determine C_2, one must know how the various functions, e.g. p_i''' depend upon the couplings. The p_i''s are defined in (3.9) and (3.10) in terms of parameters V, ζ, and ℓ $(0 \le \ell \le 1)$.

$$
P_i^{(N)} \equiv \frac{\partial^N}{\partial V^N} P(V, \zeta, \ell) \Big|_{V=\zeta} . \qquad \text{(A.31)}
$$

From the definitions one can compute a compendium of required functions in terms of ζ and ℓ.

$$
P_1' = \frac{dn(2\zeta,\ell)}{sn(2\zeta,\ell)} , \qquad P_2' = \frac{cn(2\zeta,\ell)}{sn(2\zeta,\ell)}
$$

$$
P_3' = \frac{1}{sn(2\zeta,\ell)} , \qquad P_4' = \frac{cn(2\zeta,\ell) + dn(2\zeta,\ell) - 1}{sn(2\zeta,\ell)} \qquad \text{(A.32)}
$$

$$
P_1'' = -\ell^2 cn^2(\zeta,\ell) = 1 - \ell^2 - dn^2(\zeta,\ell) \qquad \text{(A.33)}
$$

$$P_2'' = -dn^2(\zeta,\ell) \tag{A.34}$$

$$P_3'' = -\ell^2 sn^2(\zeta,\ell) = dn^2(\zeta,\ell) - 1 \tag{A.35}$$

$$P_4'' = P_1'' + P_2'' - P_3'' = 2 - \ell^2 - 3\, dn^2(\zeta,\ell) \tag{A.36}$$

$$P_1''' = \frac{1}{2}\left(-4 + 4\ell^2 + 6\, dn^2(\zeta,\ell) - \ell^2 dn^2(\zeta,\ell) - 5\, dn^4(\zeta,\ell) + dn^6(\zeta,\ell)\right) \tag{A.37}$$

$$P_2''' = \frac{1}{2}\left(-4 + 4\ell^2 + 4\, dn^2(\zeta,\ell) - \ell^2 dn^2(\zeta,\ell) + dn^4(\zeta,\ell) - dn^6(\zeta,\ell)\right) \tag{A.38}$$

$$P_3''' = \frac{1}{2}\left(6 - 4\ell^2 - 8\, dn^2(\zeta,\ell) - \ell^2 dn^2(\zeta,\ell) + dn^4(\zeta,\ell) - dn^6(\zeta,\ell)\right) \tag{A.39}$$

Notice that there are only two parameters ζ, ℓ which define the three quantities p_1', p_2', p_3'. One can see that there is an implicit restriction

$$P_3'^2 = P_2'^2 + 1 \tag{A.40}$$

Also,

$$|P_3'| \geq |P_1'| \geq |P_2'| \tag{A.41}$$

If one wants to relate a Hamiltonian with given couplings,

$$H_{xyz} = -\frac{1}{2}\sum_j \left(J_x\, \sigma_j^1 \sigma_{j+1}^1 + J_y\, \sigma_j^2 \sigma_{j+1}^2 + J_z\, \sigma_j^3 \sigma_{j+1}^3\right) \tag{A.42}$$

to

$$G_0 = \frac{1}{2}\sum_j \sum_{i=1}^{4} P_i'\, \sigma_j^i \sigma_{j+1}^i \tag{A.43}$$

up to an added constant, one must first juggle the labelling of directions X,Y,Z so that $-J_Z > -J_X > |J_Y|$ (one may also negate two of the couplings if necessary through use of a canonical transformation). Then the couplings must be scaled by an overall factor k^{-1} (see 3.15) such that for $J_i' = k^{-1} J_i$,

$$J_3'^2 = J_2'^2 + 1 . \tag{A.44}$$

One may then equate

$$p_i' = -J_i' \quad i = 1, 2, 3 \quad (J_1 \equiv J_x , J_2 \equiv J_y , J_3 \equiv J_z) \tag{A.45}$$

One can solve for ℓ^2 and dn (ζ,ℓ) in terms of J_i':

$$\ell^2 = \frac{1 - dn^2(\zeta,\ell)}{sn^2(\zeta,\ell)} = \frac{J_z'^2 - J_x'^2}{J_z'^2 - J_y^2} = J_z'^2 - J_x'^2 \tag{A.46}$$

$$dn(\zeta,\ell) = \left(\frac{J_x' + J_y'}{J_y' + J_z'}\right)^{1/2} = (J_x' + J_y')^{1/2} (J_z' - J_y')^{1/2} . \tag{A.47}$$

Then all of the functions (A.32)-(A.39) can be expressed in terms of the J_i, so one then has C_2 (A.30) for a given Hamiltonian. One can, of course, write out this form explicitly, but very little cancellation occurs and one is left with rather lengthy polynominal coefficients of up to sixth order in the J's.

Finally, setting $J_2 = 0$ one obtains

$$J_3' = 1 \tag{A.48}$$

$$\ell^2 = 1 - J_1'^2 \tag{A.49}$$

$$dn^2(\zeta,\ell) = J_1' . \tag{A.50}$$

A rather remarkable simplification then occurs. The second term of C_2 vanishes and the long coefficient of the third term reduces to $p_1' p_3'^2$ for i=1, zero for i=2, and $p_1'^2 p_3'$ for i=3. Thus for the XZ model

$$C_2 = \sum_J \left(P_1'^2 P_3' \sigma_J^1 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^1 + P_1' P_3'^2 \sigma_J^3 \sigma_{J+1}^2 \sigma_{J+2}^2 \sigma_{J+3}^3 \right.$$
$$\left. + P_1'^2 P_3' \sigma_J^3 \sigma_{J+1}^3 + P_1' P_3'^2 \sigma_J^1 \sigma_{J+1}^1 \right) \tag{A.51}$$

which is the same as the charge found earlier in Chap. V (5.10). Thus, at least for the case of C_2 the simplified procedures of Chaps. V and VII yield the same charge as that given by the original generating function (3.22).

Appendix B. Relationship Between the Ising and XZ Models

It has been known for some time that the partition function for a restricted version of the 8-vertex model can be written as the product of the partition functions of two separate, identical, two-dimensional Ising models.[29] It has also been shown that this restriction has the effect of limiting the associated τ-continuum quantum Hamiltonian to that of an XY (or XZ) model.[30] Since the one-dimensional quantum Ising model in a transverse magnetic field is the τ-continuum quantum Hamiltonian associated with the two-dimensional Ising model one might expect that there is also a correspondence between the XZ model and the sum of two one dimensional quantum Ising models. Such a correspondence might be used to translate results from one model to the other.

The suggested correspondence does indeed exist. The transformation which accomplishes it is the familiar dual transformation for a single Ising model (see sec. 4.2). For a chain of N sites it is given by

$$\mu_J^1 = \sigma_J^3 \sigma_{J+1}^3 , \quad J \neq N ; \quad \mu_N^1 = \sigma_N^3 \tag{B.1a}$$

$$\mu_J^3 = \prod_{K=1}^{J} \sigma_K^1 \tag{B.1b}$$

$$\mu_J^2 = -\left(\prod_{K=1}^{J-1} \sigma_K^1\right) \sigma_J^2 \sigma_{J+1}^3 , \quad J \neq N ; \quad \mu_N^2 = -\left(\prod_{K=1}^{N-1} \sigma_K^1\right) \sigma_N^2 \tag{B.1c}$$

Its inverse is given by

$$\sigma_J^1 = \mu_{J-1}^3 \mu_J^3 \quad J \neq 1 ; \quad \sigma_1^1 = \mu_1^3 \tag{B.2a}$$

$$\sigma_J^3 = \prod_{K=J}^{N} \mu_K^1 \tag{B.2b}$$

$$\sigma_J^2 = -\mu_{J-1}^3 \mu_J^2 \left(\prod_{K=J+1}^{N} \mu_K^1\right) , \quad J \neq 1 ; \quad \sigma_1^2 = -\mu_1^2 \left(\prod_{K=2}^{N} \mu_K^1\right) \tag{B.2c}$$

The somewhat asymmetrical appearance at the N^{th} and 1^{st} site is necessary in order for the dual variables μ_j^i to have the same commutation relations as the original variables, when one takes into account the periodic boundary conditions. If this transformation (B.2) is applied to the XZ model Hamiltonian the result is:

$$H_{XZ} \equiv -\frac{1}{2}\sum_{J=1}^{N}\left(J_x\, \sigma_J^1 \sigma_{J+1}^1 + J_z\, \sigma_J^3 \sigma_{J+1}^3\right)$$

$$= -\frac{1}{2}\sum_{J=2}^{N-1}\left(J_x\, \mu_{J-1}^3 \mu_{J+1}^3 + J_z\, \mu_J^1\right) - \frac{1}{2} J_x\, \mu_2^3$$

$$-\frac{1}{2} J_x\, \mu_{N-1}^3 \mu_N^3 \mu_1^3 - \frac{1}{2} J_z\, \mu_1^1 - \frac{1}{2} J_z \left(\prod_{J=1}^{N-1} \mu_J^1\right). \qquad \text{(B.3)}$$

Therefore, except for boundary terms, H_{XZ} is equivalent to

$$H_{xz} = -\frac{1}{2}\sum_{J=1}^{N}\left(J_x\, \mu_{J-1}^3 \mu_{J+1}^3 + J_z\, \mu_J^1\right)$$

$$= -\frac{1}{2}\sum_{J\,even}\left(J_x\, \mu_J^3 \mu_{J+2}^3 + J_z\, \mu_J^1\right) - \frac{1}{2}\sum_{J\,odd}\left(J_x\, \mu_J^3 \mu_{J+2}^3 + J_z\, \mu_J^1\right)$$

$$= -\frac{1}{2}\sum_{K=1}^{N/2}\left(J_x\, n_K^3 n_{K+1}^3 + J_z\, n_K^1\right) - \frac{1}{2}\sum_{K=1}^{N/2}\left(J_x\, \rho_K^3 \rho_{K+1}^3 + J_z\, \rho_K^1\right)$$

$$n_K^i \equiv \mu_{2K}^i \; ; \; \rho_K^i \equiv \mu_{2K-1}^i \qquad \text{(B.4)}$$

which is the sum of two separate Ising models, one on even-numbered and the other on odd-numbered lattice sites. Thus it is not surprising that, as originally discovered by Pfeuty,[31] the excitation spectrum of an Ising model

$$H_I = -\sum_{J=1}^{N/2}\left(K\, \sigma_J^3 \sigma_{J+1}^3 + \Gamma \sigma_J^1\right) \qquad \text{(B.5)}$$

with $K = \frac{1}{2} J_X$ and $\Gamma = \frac{1}{2} J_Z$ is the same as the spectrum of an XZ model[32] with twice as many sites (the XZ model has an additional two-fold degeneracy). When conserved charges for the XZ model (5.18, 5.19) are subjected to the transformation (B.2) and the boundary terms discarded, one finds that the resulting charges also split up into operators involving only even-numbered or only odd-numbered lattice sites. This property is not an automatic feature of the transformation. It can be

easily checked that each one of these sets of charges commutes with its corresponding single Ising model Hamiltonian (defined in terms of η^i_K or ρ^i_K in B.4). Thus through this transformation one can find an infinite set of conserved charges for the Ising model.[22] The resulting even numbered charges are the same charges as have been derived previously (8.9) from the theorem of Chap. VII. As far as I am aware, this transformation has not been previously reported in the literature. After publication of the result, however, I learned that it had been privately known by at least one other researcher.[33]

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