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necessary to take for α and β the appropriate linear combinations of the solutions (4.7) of Eq. (4.6). It is obvious that the computation of α and β presents formidable numerical difficulties even though the convergence of the series in Eq. (4.7) is rapid for $(B^2/\omega_0) > 10^{-4}$ webers^2-sec/meters^4. There are two special cases for which the computational labor can be reduced by the use of approximate solutions of Eq. (4.6). If $c^2 \ll |ic+b^2|$, the term in c^2 in Eq. (4.6) may be neglected, and the approximate solutions

$$w_1(z) = \cos qz \quad \text{and} \quad w_2(z) = (1/q)\sin qz, \tag{4.9}$$

where $q^2 = n + \frac{1}{2}$, may be used. If $|c|$ is large and $|b/c|$ is small, asymptotic series[4] for the Weber functions may be used.

[4] Reference 3, p. 184; E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, reprinted 1950), fourth edition, pp. 347–349.

Canonical Field Theory—A Prototype Example*

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The equations of a field may be put into a standard "Lagrangian" form from which several conservation laws follow directly. As an illustrative example, a string free to vibrate in two directions is investigated; this example clearly illustrates the outstanding features of the canonical theory, while avoiding the notational and physical complications encountered in most systems of practical interest. The conservation laws are interpreted for the string. The theory is further developed to express the field's behavior in terms of canonical coordinates and momenta. Quantum conditions are introduced, as in meson theory and quantum electrodynamics. It is shown that the mathematics of the "quantized string" is that of several charged particles occupying a set of energy states.

IN exploring the behavior of a system of particles, we commonly use two different techniques. The first is the specific approach: We discover the details of our system's behavior by considering the details of its construction. The second is the canonical approach: The system is characterized by a Lagrangian function and the details of behavior are obtained by operating on this function in standard ways. These techniques are complementary; the one is intuitive and emphasizes the peculiarities of the individual system, the other is formal and emphasizes underlying uniformities.

Both methods are adaptable to the study of continuous systems, but here the advantages of the canonical technique are not often so widely exploited. It is the purpose of this paper to develop the behavior of a taut string as an illustrative example of the canonical technique for continuous systems.

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The canonical formulation is based upon the possibility of constructing a Lagrangian function L depending on the dynamical variables of the system, such that L fulfills Hamilton's principle

$$\delta \int_{t_0}^{t_1} L dt = 0, \tag{1}$$

when we demand that our dynamical variables satisfy, as mathematical functions of time, the relationships which they actually satisfy in nature. The basic dynamical variables of a continuous system are its "field components," which we will denote by ψ_σ, functions of space coordinates and time. The Lagrangian may be expressed as

$$L = \int_R \mathfrak{L} dR, \tag{2}$$

where R is the region occupied by the system, and $\mathfrak{L} = \mathfrak{L}(\psi_1, \dot{\psi}_1, \partial\psi_1/\partial x_1, \cdots, \psi_2, \cdots, x_1, \cdots, t)$ is the "Lagrangian density" at a given point.

According to the familiar procedures of variational analysis,[1] Hamilton's principle is fulfilled if the field components satisfy the Euler equations

$$\frac{\partial \mathcal{L}}{\partial \psi_\sigma} - \sum_i \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma} = 0. \quad (3)$$

The Lagrangian density $\mathcal{L}$ must be so chosen that Eq. (3) will be the equations of motion for our system, a task which usually is not very difficult. Because these equations are homogeneous in $\mathcal{L}$, we may give $\mathcal{L}$ the dimensions of energy density without loss of generality.

Now we may state a number of conservation laws. We define a "Hamiltonian density"

$$\mathcal{H} = \sum_\sigma \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma}\dot{\psi}_\sigma - \mathcal{L} \quad (4)$$

and likewise an "energy current" vector with components

$$S_i = \sum_\sigma \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)}\dot{\psi}_\sigma. \quad (5)$$

Hamiltonian density $\mathcal{H}$ may be regarded as the energy density of the system, as it satisfies the differential conservation law $\partial\mathcal{H}/\partial t + \mathrm{div} S = 0$; for

$$\frac{\partial \mathcal{H}}{\partial t} + \sum_i \frac{\partial S_i}{\partial x_i} = \sum_\sigma \dot{\psi}_\sigma\left(\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma} + \sum_i \frac{\partial}{\partial x_i}\frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)} - \frac{\partial \mathcal{L}}{\partial \psi_\sigma}\right) + \left[\sum_\sigma\left(\frac{\partial \mathcal{L}}{\partial \psi_\sigma}\dot{\psi}_\sigma + \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma}\ddot{\psi} + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)}\frac{\partial \dot{\psi}_\sigma}{\partial x_i}\right) - \frac{\partial \mathcal{L}}{\partial t}\right] = 0. \quad (6)$$

The first term vanishes because of Eq. (3), and the second identically if $\mathcal{L}$ is not an explicit function of t. If $\mathcal{L}$ were to depend explicitly on t, the second term would not vanish and the equation would express the transfer of energy between our field and some coexistent system.

[1] See, for example, W. F. Osgood, *Advanced Calculus* (Macmillan Company, New York, 1925) Chap. 17, Sec. 6.

On the same pattern let us define a "momentum density" vector

$$G_k = -\sum_\sigma \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\sigma}\frac{\partial \psi_\sigma}{\partial x_k} \quad (7)$$

and a "stress tensor"

$$T_{ik} = -\sum_\sigma \frac{\partial \mathcal{L}}{\partial(\partial\psi_\sigma/\partial x_i)}\frac{\partial \psi_\sigma}{\partial x_k} - \mathcal{L}\delta_{ik}, \quad (8)$$

where the familiar "Kronecker delta" is 1 or 0 depending on its subscripts. Proceeding much as in Eq. (6), we may demonstrate that

$$\frac{\partial G_k}{\partial t} + \sum_i \frac{\partial T_{ik}}{\partial x_i} = 0, \quad (9)$$

provided $\mathcal{L}$ is not an explicit function of x_k. The "stress tensor" characterizes the flux of momentum through the system.

Certain cases lead to yet another differential conservation law. We can form new field variables out of combinations of the old, and these will satisfy Euler equations of the same form, because of the invariance of Hamilton's principle. Consider the case of only two field variables; form a linear combination by multiplying the first with a real and the second with a pure imaginary constant. This combination, say ψ, and its complex conjugate ψ^* determine the two original field components. Equations (4) through (9) will still hold, with the summation extending over ψ and ψ^*. This complex representation has particular virtues in the special case where $\mathcal{L}$ is invariant under transformations which change only the phase of the complex variables. That is, letting $\psi = \psi_0 e^{i\alpha}$,

$$\mathcal{L}(\psi_0, \psi_0^*, \partial\psi_0/\partial x_1, \cdots) = \mathcal{L}(\psi, \psi^*, \partial\psi/\partial x_1, \cdots) = \mathcal{L}(\psi_0 e^{i\alpha}, \psi_0^* e^{-i\alpha}, \partial\psi_0/\partial x_1 e^{i\alpha}, \cdots).$$

Equivalently, $d\mathcal{L}/d\alpha = 0$. Now

$$d\psi/d\alpha = i\psi,\ d\psi^*/d\alpha = -i\psi^*,$$

$$\frac{d}{d\alpha}\frac{\partial \psi}{\partial x_1} = i\frac{\partial \psi}{\partial x_1}, \cdots,$$

so that

$$\frac{d\mathcal{L}}{d\alpha}=i\left[\frac{\partial\mathcal{L}}{\partial\psi}\psi-\frac{\partial\mathcal{L}}{\partial\psi^*}\psi^*+\frac{\partial\mathcal{L}}{\partial\dot\psi}\dot\psi-\frac{\partial\mathcal{L}}{\partial\dot\psi^*}\dot\psi^*+\sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\frac{\partial\psi}{\partial x_i}-\frac{\partial\mathcal{L}}{\partial(\partial\psi^*/\partial x_i)}\frac{\partial\psi^*}{\partial x_i}\right)\right]=0. \quad (10)$$

The vanishing of this last expression leads us to the additional conservation law. We consider the scalar

$$\rho=-i\epsilon\left(\frac{\partial\mathcal{L}}{\partial\dot\psi}\psi-\frac{\partial\mathcal{L}}{\partial\dot\psi^*}\psi^*\right) \quad (11)$$

and the vector

$$\sigma_i=-i\epsilon\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\psi-\frac{\partial\mathcal{L}}{\partial(\partial\psi^*/\partial x_i)}\psi^*\right). \quad (12)$$

Then with the aid of Eqs. (3) and (10),

$$\frac{\partial\rho}{\partial t}+\sum_i\frac{\partial\sigma_i}{\partial x_i}=-i\epsilon\left[\frac{\partial\mathcal{L}}{\partial\psi}\psi-\frac{\partial\mathcal{L}}{\partial\psi^*}\psi^*+\frac{\partial\mathcal{L}}{\partial\dot\psi}\dot\psi-\frac{\partial\mathcal{L}}{\partial\dot\psi^*}\dot\psi^*+\sum_i\left(\frac{\partial\mathcal{L}}{\partial(\partial\psi/\partial x_i)}\frac{\partial\psi}{\partial x_i}-\frac{\partial\mathcal{L}}{\partial(\partial\psi^*/\partial x_i)}\frac{\partial\psi^*}{\partial x_i}\right)\right]=0. \quad (13)$$

In the canonical terminology the scalar ρ and the vector σ are called the "electric charge density" and the "electric current" and ϵ is chosen to make their dimensions fit.[2]

Now for the taut string. Let μ and τ be its constant linear density and tension. Distance along the string we will denote by x. Because x is the only spatial dimension of the system, the summation over i in the formulas above reduces to a single term. The string may be given two independant displacements at right angles to its length, ξ and η, our "field components." It is reasonably evident that the kinetic and potential energy densities are $\frac{1}{2}\mu(\dot\xi^2+\dot\eta^2)$ and $\frac{1}{2}\tau[(\partial\xi/\partial x)^2+(\partial\eta/\partial x)^2]$, provided the derivatives are not too large; these expressions are incidental to our subsequent development.

The first step of the canonical formalism is to concoct a Lagrangian density which will yield the proper equations of motion. It is natural to try the difference between kinetic and potential energies, so we set

$$\mathcal{L}=\tfrac{1}{2}\mu(\dot\xi^2+\dot\eta^2)-\tfrac{1}{2}\tau\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2\right]. \quad (14)$$

The Euler equations reduce to

$$\frac{\partial\mathcal{L}}{\partial\eta}-\frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial\eta/\partial x)}-\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot\eta}=0,$$

or

$$\tau\frac{\partial^2\eta}{\partial x^2}-\mu\ddot\eta=0 \quad (3a)$$

and an exactly similar equation for ξ. This is of course just the familiar wave equation for a taut string, and justifies our choice of $\mathcal{L}$.

We are now in a position to specialize the various expressions given earlier.

Canonical energy density:

$$\mathcal{H}=\frac{\partial\mathcal{L}}{\partial\dot\xi}\dot\xi+\frac{\partial\mathcal{L}}{\partial\dot\eta}\dot\eta-\mathcal{L}=\tfrac{1}{2}\mu(\dot\xi^2+\dot\eta^2)+\tfrac{1}{2}\tau\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2\right]. \quad (4a)$$

Canonical energy current:

$$S=\frac{\partial\mathcal{L}}{\partial(\partial\xi/\partial x)}\dot\xi+\frac{\partial\mathcal{L}}{\partial(\partial\eta/\partial x)}\dot\eta=-\tau\left(\frac{\partial\xi}{\partial x}\dot\xi+\frac{\partial\eta}{\partial x}\dot\eta\right). \quad (5a)$$

Canonical momentum density:

$$G=-\frac{\partial\mathcal{L}}{\partial\dot\xi}\frac{\partial\xi}{\partial x}-\frac{\partial\mathcal{L}}{\partial\dot\eta}\frac{\partial\eta}{\partial x}=-\mu\left(\dot\xi\frac{\partial\xi}{\partial x}+\dot\eta\frac{\partial\eta}{\partial x}\right). \quad (7a)$$

[2] This whole development has been adapted from G. Wentzel's *Quantum Theory of Fields* (Interscience Publications, New York, 1949).

Canonical stress tensor:

$$T=-\frac{\partial\mathcal{L}}{\partial(\partial\xi/\partial x)}\frac{\partial\xi}{\partial x}-\frac{\partial\mathcal{L}}{\partial(\partial\eta/\partial x)}\frac{\partial\eta}{\partial x}+\mathcal{L}$$
$$=\tfrac{1}{2}\tau\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial x}\right)^2\right]+\tfrac{1}{2}\mu(\dot{\xi}^2+\dot{\eta}^2). \quad (8a)$$

These equations are open to easy physical interpretation. There can be little doubt that $\mathcal{H}$ is an energy density in good standing. S satisfies the conservation law (6) jointly with $\mathcal{H}$, and so is a *bona fide* energy current; more simply, S is the inner product of transverse tension and velocity, hence the flux of energy. According to Eq. (5a) there can be no energy current flowing at the fixed ends of the string where $\dot{\xi}=\dot{\eta}=0$. The integral of $\mathcal{H}$ over the whole string, the total energy of the system, is constant in time, because of Eq. (6). The canonical momentum density G of Eq. (7a) is not so clear-cut a case. The physical set up forbids actual momentum in the x direction. We note that $G=(\mu/\tau)S$ and according to the wave equation (3a) the constant $\mu/\tau=1/v^2$ is the inverse square of the string's propagation velocity. The situation is reminiscent of that in electromagnetic theory, where the momentum density of the field is proportional to the energy flux and inversely proportional to the squared velocity of light. Equation (8a) shows that the flux of canonical momentum is equal to the energy density. According to Eq. (8a), in general, T need not vanish at the string's ends. However, so long as $\partial\xi/\partial x$ and $\partial\eta/\partial x$ vanish in the end regions, the integral of G over the whole string will be constant in time as may be seen from Eq. (9). Thus the integral of energy current $S=v^2G$ will likewise remain constant until the disturbance reaches an end of the string, a situation we would hardly have looked for, had it not been for our canonical theory.

The axial symmetry of the system suggests a further exploration. Let

$$\psi=(\xi+i\eta)/\sqrt{2}, \quad (15a)$$

so that

$$\xi=(\psi+\psi^*)/\sqrt{2}, \quad \eta=-i(\psi-\psi^*)/\sqrt{2}. \quad (15b)$$

The Lagrangian density becomes

$$\mathcal{L}=\mu\dot{\psi}\dot{\psi}^*-\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x} \quad (14a)$$

and the Euler equation for ψ,

$$\tau\frac{\partial^2\psi}{\partial x^2}-\mu\ddot{\psi}=0. \quad (3b)$$

Equations (4a), (5a), (7a), and (8a) become

$$\mathcal{H}=\mu\dot{\psi}\dot{\psi}^*+\tau\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x} \quad (4b)$$

$$S=-\tau\left(\frac{\partial\psi^*}{\partial x}\dot{\psi}+\frac{\partial\psi}{\partial x}\dot{\psi}^*\right) \quad (5b)$$

$$G=-\mu\left(\frac{\partial\psi^*}{\partial x}\dot{\psi}+\frac{\partial\psi}{\partial x}\dot{\psi}^*\right) \quad (7b)$$

$$T=\mu\dot{\psi}\dot{\psi}^*+\tau\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x}. \quad (8b)$$

The Lagrangian (14a) is evidently invariant under phase changes in ψ. Thus we have canonical electric charge density:

$$\rho=-i\epsilon\mu(\dot{\psi}^*\psi-\dot{\psi}\psi^*). \quad (11a)$$

Canonical electric current:

$$\sigma=-i\epsilon\tau\left(\frac{\partial\psi}{\partial x}\psi^*-\frac{\partial\psi^*}{\partial x}\psi\right). \quad (12a)$$

Our dislocated terminology does not mean that these expressions are fantasies, however. Substituting Eq. (15a) into Eqs. (11a) and (12a),

$$\rho=\epsilon\mu(\dot{\xi}\eta-\dot{\eta}\xi),$$
$$\sigma=\epsilon\tau\left(\xi\frac{\partial\eta}{\partial x}-\eta\frac{\partial\xi}{\partial x}\right).$$

Aside from the unfortunate dimensional coefficient, ρ is the angular momentum density of the system. We might call σ the "torque potential" as ϵ^{-1} times its space derivative

$$\frac{\partial\sigma}{\partial x}=\epsilon\tau\left(\xi\frac{\partial^2\eta}{\partial x^2}-\eta\frac{\partial^2\xi}{\partial x^2}\right)$$

is evidently the torque density acting about the axis of the system. Thus our final conservation law in this case simply states that the angular acceleration of a bit of string is proportional to the torque exerted upon it. At the end points σ must vanish with ψ so that the integral of ρ, and the total angular momentum, will be constant in time, as may be seen from Eq. (13).

The canonical field theory can be brought even closer to that of particles. The field function ψ may be regarded as a set of canonical coordinates in the ordinary mechanical sense, one for each point x. With the string fixed at two points, $x=0$ and $x=l$, we can make a linear transformation to a new set of coordinates,

$$q_n=\frac{\sqrt{2}}{l}\int_0^l \psi \sin\lambda_n x dx, \quad \text{where} \quad \lambda_n=\frac{n\pi}{l} \tag{16a}$$

whose inverse transformation is

$$\psi=\sqrt{2}\sum_n q_n \sin\lambda_n x. \tag{16b}$$

We will call the infinite set of q's "modes of motion." Equation (16b) may now be used with Eq. (2) to express the integrated Lagrangian in terms of new arguments,

$$L=\int_0^l \mathcal{L}(\psi, \dot{\psi}, \partial\psi/\partial x, \cdots)dx$$
$$=L(q_1, \dot{q}_1, q_2, \dot{q}_2, \cdots). \tag{17}$$

Hamilton's principle (1) is equivalent to the Euler equations

$$(d/dt)(\partial L/\partial \dot{q}_n)-(\partial L/\partial q_n)=0, \quad n=1, 2, \tag{18}$$

which are just Lagrange's equations of motion. With very minor notational elaborations this development may be made as general as that beginning this paper.

Now to apply this new machinery. We first notice that by Eq. (16b)

$$\frac{\partial\psi}{\partial x}=\sqrt{2}\sum_n \lambda_n q_n \cos\lambda_n x,$$

(provided ψ is a physically permissible function). From (17) the total Lagrangian of our string system is

$$L=\int_0^l\left(\mu\dot{\psi}\dot{\psi}^*-\tau\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x}\right)dx$$
$$=2\mu\int_0^l (\sum_n \dot{q}_n \sin\lambda_n x)$$
$$\times(\sum_m \dot{q}_m^* \sin\lambda_m x)dx$$
$$-2\tau\int_0^l (\sum_n \lambda_n q_n \cos\lambda_n x)$$
$$\times(\sum_m \lambda_m q_m^* \cos\lambda_m x)dx$$
$$=\sum_n l\mu\dot{q}_n\dot{q}_n^*-l\tau\lambda_n^2 q_n^2 q_n^*, \tag{17a}$$

where the last step follows from the orthogonality of the various functions. Putting this Lagrangian into Eq. (18) (and taking conjugates for notational convenience), we get the equations of motion

$$l\mu\ddot{q}_n+l\tau\lambda_n^2 q_n=0 \quad \text{or} \quad \ddot{q}_n+v^2\lambda_n^2 q_n=0, \tag{18a}$$

which integrate at once to

$$q_n=q_n^0 \cos\lambda_n vt+(\lambda_n v)^{-1}\dot{q}_n^0 \sin\lambda_n vt.$$

This result may now be substituted back into Eq. (16b). Given the initial state of our system we can determine the constants q_n^0 and $\dot{q}_n^0$ from Eq. (16a). Substituting back into Eq. (15b) we obtain ξ and η as explicit functions of position and time; and the full solution of the string problem.

We may easily express the total energy and "charge" in terms of the modes of motion of the system. Proceeding exactly as in Eq. (17a), we find

$$H=\int_0^l \mathcal{H}dx=\sum_n l\mu\dot{q}_n\dot{q}_n^*+l\tau\lambda_n^2 q_n q_n^* \tag{19}$$

$$Q=\int_0^l \rho dx=-i\epsilon l\mu\sum_n \dot{q}_n^* q_n-q_n^*\dot{q}_n. \tag{20}$$

Having come so far, it is practical to take the field theory yet a step further, and bring it into "Hamiltonian form." With every mode coordinate q_n we associate a conjugate momentum

$$p_n=\partial L/\partial\dot{q}_n. \tag{21}$$

From Eq. (17a) we get $p_n=l\mu\dot{q}_n^*$. Equations (19) and (20) become

$$H=\sum_n\frac{1}{l\mu}p_n^* p_n+l\tau\lambda_n^2 q_n q_n^* \tag{19a}$$

$$Q=-i\epsilon\sum_n p_n q_n-p_n^* q_n^*. \tag{20a}$$

The Hamiltonian (19) determines the equations of motion (18a) by means of Hamilton's canonical equations $\dot{q}_n=\partial H/\partial p_n$, $\dot{p}_n=-\partial H/\partial q_n$ as may be quickly checked. For brevity's sake we omit the rather long general proof.

While the use of complex dynamical variables has made our work very compact, it has disguised some familiar features of the mechanical system. Thus we briefly revert to real coordinates x_n, y_n:

$$q_n=(x_n+iy_n)/\sqrt{2}$$

and proceeding as in Eq. (21) to find their conjugate momenta p_{xn}, p_{yn} deduce that

$$p_n=(p_{xn}-ip_{yn})/\sqrt{2}.$$

Substituting these results into Eq. (19a), we find

$$H=\sum_n\left(\frac{1}{2l\mu}p_{xn}^2+\frac{l\tau\lambda_n^2}{2}x_n^2\right)+\left(\frac{1}{2l\mu}p_{yn}^2+\frac{l\tau\lambda_n^2}{2}y_n^2\right). \quad (19b)$$

But this is just the Hamiltonian of a collection of independent simple harmonic oscillators. We can do even a little better: the contribution of the nth set of variables is the Hamiltonian of a two-dimensional isotropic oscillator with "canonical mass" $m=l\mu$, "canonical spring constant" $k_n=l\tau\lambda_n^2$ and angular frequency $\omega_n=(k_n/m)^{\frac{1}{2}}=(\tau/\mu)^{\frac{1}{2}}\lambda_n$.

The outstanding use of the general theory we have developed here is in the quantum theory of fields. While the quantized string is hardly so common as its classical counterpart, it is none the less worth investigating because of the ease with which it demonstrates general principles. Field quantization follows from postulated quantum conditions of the familiar form,

$$x_np_{xn}-p_{xn}x_n=y_np_{yn}-p_{yn}y_n=i\hbar, \quad (21)$$

while the various other combinations commute.[3] From this easily follows the condition $q_np_n-p_nq_n=i\hbar$ for the complex modes and their momenta. Because the Hamiltonian is arbitrary to the extent of an additive constant, we may subtract $\hbar\omega_n$ from the contribution of the nth mode, obtaining

$$H=\sum_n m^{-1}p_n{}^*p_n+m\omega_n{}^2q_nq_n{}^*-\hbar\omega_n \quad (19c)$$

in order to avoid a convergence difficulty later.

From here on the quantum analysis is fairly standard. If we let

$$a_n=(2m\hbar\omega_n)^{\frac{1}{2}}(p_n+im\omega_nq_n{}^*),$$
$$b_n=(2m\hbar\omega_n)^{\frac{1}{2}}(p_n{}^*+im\omega_nq_n) \quad (22a)$$

so that

$$q_n=i\left(\frac{\hbar}{2m\omega_n}\right)^{\frac{1}{2}}(a_n{}^*-b_n),$$

$$p_n=\left(\frac{m\hbar\omega_n}{2}\right)^{\frac{1}{2}}(a_n+b_n{}^*), \quad (22b)$$

it is quickly demonstrated that Eq. (21a) is equivalent to

$$a_n{}^*a_n-a_na_n{}^*=b_n{}^*b_n-b_nb_n{}^*=1 \quad (23)$$

with all other fundamental pairs commuting. The operators $N_{an}=a_na_n{}^*$ and $N_{bn}=b_nb_n{}^*$ are evidently Hermitian. In a brief and elegant proof Dirac has shown[4] that Eq. (23) insures that the eignevalues of N_{an} must be

$$N'_{an}=0, 1, 2, 3,\cdots, \quad (24)$$

and similarly for N_{bn}. An even briefer, if less compelling, argument[5] runs like this: Represent a_n and b_n as independant basic oscillator matrices, satisfying Eq. (23); then Eq. (24) follows at once. Or by forsaking the convenience of our complex modes, we may deduce the same results more laboriously from the Schrödinger equation of an isotropic oscillator.

We now substitute Eq. (22b) into Eqs. (19c) and (20a) to obtain

$$H=\sum_n (N_{an}+N_{bn})\hbar\omega_n \quad (19d)$$

and

$$Q=\sum_n \epsilon\hbar(N_{an}-N_{bn}). \quad (20b)$$

The energy and "charge" eigenstates of our system will be the simultaneous eigenstates of

[3] Field quantization may be approached from several different angles. Ours is roughly that found in Heitler's *Quantum Theory of Radiation* (Clarendon Press, Oxford, 1945).

[4] *Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), third edition, p. 136.

[5] G. Wentzel, reference 2, Sec. 8.

all the N's. The eigenvalues of H and Q may be obtained by substituting the various eigenvalues of the N's into Eqs. (19d) and (20a).

All that remains is to remark on the "canonical interpretation" of these results. The quantum string represents a system of "particles." The wave equation (3a) is the Schrödinger equation of one such particle (or better the Schrödinger-Gordon equation, as it is second order in time). If the particle is "confined in a box" by the end conditions at $x=0$ and $x=l$, according to elementary quantum mechanics it is limited to a discrete set of possible energy eigenstates, the classical string's discrete modes of motion. Upon "second quantization" the system contains several particles; the operator $N_{an}+N_{bn}$ represents the number of particles in the nth energy state, as its contribution to the total energy operator (19d) clearly shows. There are two sorts of particles present, carrying, respectively, positive and negative canonical electric charges of value $\epsilon\hbar$. In the nth energy state there are N_{an} positive and N_{bn} negative particles, as demonstrated by the contribution of the nth state to the total charge within the system [Eq. (20b)]. Finally, the formalism makes no statements at all concerning which particles are in which states; the particles satisfy the Bose-Einstein statistics.

In one respect our example differs from the ordinary situation in quantum field theory. The displacement field must vanish at the two fixed ends of the string. So to preserve the realism of our treatment we expanded the displacement field as a sine series, each term vanishing at the ends. Such boundary conditions are not common; the ordinary procedure is to expand the field as a Fourier series of complex exponentials and argue that by making the domain of the expansion much larger than the interesting region of the system, we may make boundary effects as negligible as we please. In the typical boundary-free problem the total canonical momentum is constant; this is not the case in our problem, for the stress-tensor need not vanish at the ends of the string, and canonical momentum may be transferred between the string and its end supports. This may also be seen by the fact that, were we to expand our field in exponentials, we could express the canonical momentum as the sum of contributions of the various modes, as we have done for energy and canonical charge in Eqs. (19) and (20). This is not possible in the case of the sine expansion, for in the momentum density expression sines and cosines become mixed in the same products, and we lose the crucial orthogonality property. On the basis of quantum mechanics this momentum difficulty is just what one would expect from the end conditions imposed, for in the equivalent "box problem" the energy states are not momentum eigenfunctions but rather represent particles traversing the box in both directions. The particles are reflected at the ends with the momentum change this reflection brings about.

There is a very great deal more that might be said concerning the physical situation implied by the mathematical analysis of the quantum string. We will not enter into this long story, but simply remark that its major pieces are already in our hands. For we have shown that by choosing the proper coordinates we may separate our field into a set of independent oscillators. And we know, according to basic quantum mechanics, both the physical interpretation of a single oscillator and that of a system which can be separated into subsystems of known character. This is indeed our foremost justification for investigating the quantized string: it is an excellent conceptual link, a system within the scheme of quantum mechanics and also the simplest prototype of systems dealt with by the quantum theory of fields.

Erratum: Analysis and Synthesis of Optical Images

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In Eq. (6) of the above paper, instead of $(-i/2\lambda^{\frac{1}{2}})$ read $(-i)^{\frac{1}{2}}/2\lambda^{\frac{1}{2}}$.